J. Differential Equations 252 (2012) 5875–5922

Contents lists available at SciVerse ScienceDirect

Journal of Differential Equations

www.elsevier.com/locate/jde

Sturm–Liouville boundary value problems with operator potentials and unitary equivalence

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ARTICLE INFO

Article history:
Received 17 February 2011
Revised 16 February 2012
Available online 17 March 2012

MSC:
34G10
47E05
47F05
47A20
47B25

Keywords:
Sturm–Liouville operators
Operator potentials
Elliptic partial differential operators
Boundary value problems
Self-adjoint extensions
Unitary equivalence
Direct sums of symmetric operators

ABSTRACT

Consider the minimal Sturm–Liouville operator $A = A_{\min}$ generated by the differential expression

$$\mathcal{A} := -\frac{d^2}{dt^2} + T$$

in the Hilbert space $L^2(\mathbb{R}_+, \mathcal{H})$ where $T = T^* \geqslant 0$ in $\mathcal{H}$. We investigate the absolutely continuous parts of different self-adjoint realizations of $\mathcal{A}$. In particular, we show that Dirichlet and Neumann realizations, A^D and A^N, are absolutely continuous and unitary equivalent to each other and to the absolutely continuous part of the Krein realization. Moreover, if $\inf \sigma_{\mathrm{ess}}(T) = \inf \sigma(T)$, then the part $\widetilde{A}^{ac} E_{\widetilde{A}}(\sigma(A^D))$ of any self-adjoint realization $\widetilde{A}$ of $\mathcal{A}$ is unitarily equivalent to A^D. In addition, we prove that the absolutely continuous part $\widetilde{A}^{ac}$ of any realization $\widetilde{A}$ is unitarily equivalent to A^D provided that the resolvent difference $(\widetilde{A} - i)^{-1} - (A^D - i)^{-1}$ is compact. The abstract results are applied to elliptic differential expressions in the half-space.

Contents

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0022-0396/$ – see front matter
doi:10.1016/j.jde.2012.02.018

1. Introduction

Let T be a non-negative unbounded self-adjoint operator in an *infinite* dimensional separable Hilbert space $\mathcal{H}$. We consider the minimal Sturm–Liouville operator A generated by the differential expression

$$\mathcal{A} := -\frac{d^2}{dt^2} + T \tag{1.1}$$

in the Hilbert space $\mathfrak{H} := L^2(\mathbb{R}_+, \mathcal{H})$ of $\mathcal{H}$-valued square summable vector-valued functions. Following [19,20] the minimal operator $A := A_{\min}$ is introduced to be the closure of the operator A' defined by

$$A' := \mathcal{A} \upharpoonright \mathcal{D}_0, \quad \mathcal{D}_0 := \left\{ \sum_{1 \leqslant j \leqslant k} \phi_j(t) h_j \colon \phi_j \in W_0^{2,2}(\mathbb{R}_+),\ h_j \in \mathrm{dom}(T),\ k \in \mathbb{N} \right\}, \tag{1.2}$$

where $W_0^{2,2}(\mathbb{R}_+) := \{\phi \in W^{2,2}(\mathbb{R}_+)\colon\ \phi(0) = \phi'(0) = 0\}$, that is, $A_{\min} := \overline{A'}$. It is easily seen that A is a closed non-negative symmetric operator in $\mathcal{H}$ with equal deficiency indices $n_\pm(A) = \dim(\mathcal{H})$. The adjoint operator A^* of $A = A_{\min}$ is the maximal operator denoted by $A_{\max}$. Self-adjoint extensions of A (are also called self-adjoint realizations of $\mathcal{A}$) were investigated for the first time by M.L. Gorbachuk [19] in the case of finite intervals I. He proved that the traces of vector-functions $f \in \mathrm{dom}(A_{\max})$ belong to the space $\mathcal{H}_{-1/4}(T)$, cf. (5.2) and, in particular, $\mathrm{dom}(A_{\max})$ is not contained in the Sobolev space $W^{2,2}(I, \mathcal{H})$. Based on this result he constructed a boundary triplet for the operator $A_{\max} = A^*_{\min} = A^*$ in the Hilbert space $L^2(I, \mathcal{H})$ and described all self-adjoint realizations of $\mathcal{A}$ in terms of boundary conditions. These results are similar to those for elliptic operators in domains with smooth boundaries, cf. [4,24,34], and go back to classical papers of M.I. Višik [42] and G. Grubb [23].

After the pioneering work [19] the spectral theory of self-adjoint and dissipative realizations of $\mathcal{A}$ in $L^2(I,\mathcal{H})$ has intensively been investigated by several authors for the case of bounded intervals I. Their results have been summarized in the book of M.L. Gorbachuk and V.I. Gorbachuk [20, Section 4] where one finds, in particular, discreteness criteria, asymptotic formulas for the eigenvalues, resolvent comparability results, etc. Some results from [20] including the construction of a boundary triplet were extended in [21,22,32] and [13, Section 9] to the case of semi-axis. In particular, in [21,22] the $\mathfrak{S}_p$-resolvent comparability of two realizations of the form $y'(0) = C_j y(0)$, $j \in \{1,2\}$, was investigated. For instance, Dirichlet and Neumann realizations are $\mathfrak{S}_1$-resolvent comparable if and only if $T^{-1} \in \mathfrak{S}_1$ (cf. [21]).

However neither the absolutely continuous spectrum (in short *ac*-spectrum) nor the unitary equivalence of self-adjoint realizations of $\mathcal{A}$ have been investigated in previous papers. We show, cf. Lemma 5.1, that the domain $\mathrm{dom}(A)$ of the minimal operator A coincides algebraically and topologically with the Sobolev space $W^{2,2}_{0,T}(\mathbb{R}_+,\mathcal{H}) := \{f \in W^{2,2}_T(\mathbb{R}_+,\mathcal{H})\colon\ f(0) = f'(0) = 0\}$, where $W^{2,2}_T(\mathbb{R}_+,\mathcal{H})$ consists of $\mathcal{H}$-valued functions $f(\cdot) \in W^{2,2}(\mathbb{R}_+,\mathcal{H})$ satisfying

$$\|f\|^2_{W^{2,2}_T} := \int_{\mathbb{R}_+} \bigl(\|f''(t)\|^2_{\mathcal{H}} + \|f(t)\|^2_{\mathcal{H}} + \|Tf(t)\|^2_{\mathcal{H}}\bigr)\,dt < \infty.$$

This statement is similar to the classical regularity result for minimal elliptic operators with smooth coefficients, see [4,24,34]. Besides we show that Dirichlet and Neumann realizations defined as the restrictions of A^* to the domains

$$\begin{aligned} \mathrm{dom}\bigl(A^D\bigr) &:= \bigl\{f \in W^{2,2}_T(\mathbb{R}_+,\mathcal{H})\colon\ f(0) = 0\bigr\}, \\ \mathrm{dom}\bigl(A^N\bigr) &:= \bigl\{f \in W^{2,2}_T(\mathbb{R}_+,\mathcal{H})\colon\ f'(0) = 0\bigr\}, \end{aligned} \tag{1.3}$$

are self-adjoint, cf. Proposition 5.2. This statement is similar to that of the regularity of Dirichlet and Neumann realizations in the elliptic theory (cf. [4,24,34]). It looks surprising, that these regularity statements were not obtained in previous papers even in the case of bounded intervals.

Moreover, we show that the realizations A^D and A^N are absolutely continuous and unitarily equivalent for any $T = T^* \geqslant 0$. We note that these results can easily be obtained using the tensor product structure of A^D and A^N, see Appendix A.2. However, the method fails if the special tensor product structure is missing. We investigate the spectral properties of arbitrary self-adjoint realizations of $\mathcal{A}$ by using the corresponding Weyl functions.

We point out that the results substantially differ from those for Dirichlet and Neumann extensions A^D_I and A^N_I of $\mathcal{A}$ on a bounded interval I. In the latter case *the spectral properties of A^D_I and A^N_I strongly correlate with those of T*, cf. Appendix A.1. In particular, we show that in contrast to the case of a finite interval, for any $T = T^* \geqslant 0$ none of the realizations of $\mathcal{A}$ on the semi-axis is pure point, purely singular or discrete. Moreover, we prove that for any $T = T^* \geqslant 0$ the Dirichlet and the Neumann realizations A^D and A^N are *ac*-minimal in the following sense.

Definition 1.1. (See [38, Definitions 3.5, 5.1].) Let A be a closed symmetric operator and let A_0 be a self-adjoint extension of A.

(i) We say that A_0 is *ac*-minimal if for any self-adjoint extension $\widetilde{A}$ of A the absolutely continuous part (*ac*-part) A_0^{ac} is unitarily equivalent to a part of $\widetilde{A}$.

(ii) Let $\sigma_0 := \sigma_{ac}(A_0)$. We say that A_0 is strictly *ac*-minimal if for any self-adjoint extension $\widetilde{A}$ of A the part $\widetilde{A}^{ac}E_{\widetilde{A}}(\sigma_0)$ of $\widetilde{A}$ is unitarily equivalent to the *ac*-part A_0^{ac} of A_0.

Using the concept of spectral multiplicity function $N_{\widetilde{A}}(\cdot)$ of $\widetilde{A}$, cf. Appendix B, Definition B.2, one can rewrite Definition 1.1 as follows.

(i) A_0 is *ac-minimal* if and only if for any extension $\widetilde{A} = \widetilde{A}^*$ of A the inequality $N_{\widetilde{A}^{ac}}(t) \geqslant N_{A_0^{ac}}(t)$ is valid for a.e. $t \in \mathbb{R}$.
(ii) A_0 is *strictly ac-minimal* if for any extension $\widetilde{A} = \widetilde{A}^*$ of A the equality $N_{\widetilde{A}^{ac}}(t) = N_{A_0^{ac}}(t)$ holds for a.e. $t \in \sigma_0$.

One of our main results, which follows from Theorem 5.7, Theorem 5.8 and Corollary 5.9, can be summarized as follows:

Theorem 1.2. *Let $T = T^*$ be a non-negative (not necessarily unbounded) operator in the Hilbert space $\mathcal{H}$ with $t_0 = \inf \sigma(T)$ and $t_1 = \inf \sigma_{\mathrm{ess}}(T)$. Let also $\widetilde{A}$ be a self-adjoint realization of $\mathcal{A}$. Then the following hold:*

(i) *Dirichlet and Neumann realizations A^D and A^N of $\mathcal{A}$ are unitarily equivalent, absolutely continuous $\sigma_{ac}(A^D) = \sigma(A^N) = \sigma_{ac}(A^N) = [t_0, \infty)$.*
(ii) *Dirichlet, Neumann and Krein realizations A^D, A^N and A^K of $\mathcal{A}$, respectively, are ac-minimal.*
(iii) *These realizations are strictly ac-minimal if and only if $t_0 = t_1$.*
(iv) *If, in addition, either*

$$(\widetilde{A} - i)^{-1} - \left(A^D - i\right)^{-1} \in \mathfrak{S}_\infty(\mathfrak{H}) \quad \textit{or} \quad (\widetilde{A} - i)^{-1} - \left(A^K - i\right)^{-1} \in \mathfrak{S}_\infty(\mathfrak{H}),$$

then the ac-part $\widetilde{A}^{ac}$ of $\widetilde{A}$ is unitarily equivalent to the Dirichlet realization A^D.
(v) *If $t_0 = t_1$, then the ac-part $\widetilde{A}^{ac}$ of $\widetilde{A}$ is unitarily equivalent to the Dirichlet realization A^D provided that*

$$(\widetilde{A} - i)^{-1} - \left(A^N - i\right)^{-1} \in \mathfrak{S}_\infty(\mathfrak{H}).$$

At first glance it seems that the *ac*-minimality of A^D contradicts the classical Weyl–von Neumann theorem, cf. [25, Theorem X.2.1], which guarantees the existence of a Hilbert–Schmidt perturbation $C = C^*$ such that the spectrum $\sigma(A^D + C)$ of the perturbed operator $A^D + C$ is pure point. In fact, Theorem 1.2 presents an explicit example showing that the analog of the Weyl–von Neumann theorem does not hold for non-additive classes of perturbations. Indeed, Theorem 1.2 shows that for the class of self-adjoint extensions of A the absolutely continuous part can never be eliminated. Moreover, if $(\widetilde{A} - i)^{-1} - (A^D - i)^{-1}$ is compact, then even unitary equivalence holds.

We apply Theorem 1.2 and other abstract results to Schrödinger operators

$$\mathcal{L} := -\frac{\partial^2}{\partial t^2} - \sum_{j=1}^{n} \frac{\partial^2}{\partial x^2} + q(x) = -\frac{\partial^2}{\partial t^2} - \Delta_x + q, \quad (t, x) \in \mathbb{R}_+ \times \mathbb{R}^n,$$

considered in the half-space $\mathbb{R}^{n+1}_+ = \mathbb{R}_+ \times \mathbb{R}^n$, $n \in \mathbb{N}$. Here q is a bounded non-negative potential, $q = \bar{q} \in L^\infty(\mathbb{R}^n)$, $q \geqslant 0$. In this case the minimal elliptic operator $L := L_{\min}$ generated in $L^2(\mathbb{R}^{n+1}_+)$ by the differential expression $\mathcal{L}$ can be identified with the minimal operator $A = A_{\min}$ generated in $\mathfrak{H} = L^2(\mathbb{R}_+, \mathcal{H})$, $\mathcal{H} := L^2(\mathbb{R}^n)$, by the differential expression (1.1) with $T = -\Delta_x + q = T^*$. Therefore and due to the regularity theorem (see [24,34]) the Dirichlet L^D and the Neumann L^N realizations of the elliptic expression $\mathcal{L}$ are identified, respectively, with the realizations A^D and A^N of the expression $\mathcal{A}$. Moreover, the Krein realization L^K of $\mathcal{L}$ is identical with A^K. This leads to statements on realizations of $\mathcal{L}$ which are similar to those of Theorem 1.2. In fact, one has only to replace A by L in Theorem 1.2. In addition, if the condition

$$\lim_{|x| \to \infty} \int\limits_{|x-y| \leqslant 1} q(y)\, dy = 0 \tag{1.4}$$

is satisfied, then L^D and L^N are absolutely continuous and strictly *ac*-minimal. In particular, $\sigma(L^D) = \sigma_{ac}(L^D) = \sigma(L^N) = \sigma_{ac}(L^N) = [0, \infty)$.

We investigate self-adjoint realizations of $\mathcal{A}$ in the framework of extension theory. More precisely, we apply the boundary triplet approach to extensions of symmetric operators, a tool that has been intensively elaborated during the last three decades (see for instance [13,14,20] or [11] and references therein). The key role in this theory plays the so-called abstract Weyl function introduced and investigated in [13,14,12]. Moreover, the proofs invoke techniques elaborated in [2,10] and our recent publication [38].

Namely, the proofs of unitary equivalence are based on some statements from [38] and [35], which allow to compute the spectral multiplicity function $N_{\widetilde{A}^{ac}}(\cdot)$ of the *ac*-part $\widetilde{A}^{ac}$ of an extension $\widetilde{A} = \widetilde{A}^*$ in terms of boundary values of the Weyl functions at the real axis, cf. Proposition 2.8 and Corollary 2.9.

We construct a special boundary triplet for the operator A^* (in the case of unbounded $T = T^* \geqslant 0$) representing A as a direct sum of minimal Sturm–Liouville operators S_n with bounded operator potentials $T_n := TE_T([n-1, n))$, $n \in \mathbb{N}$, where $E_T(\cdot)$ is the spectral measure of T. The corresponding Weyl function $M(\cdot)$ has weak boundary values

$$M(\lambda) := M(\lambda + i0) = \operatorname*{w-lim}_{y\downarrow 0} M(\lambda + iy) \quad \text{for a.e. } \lambda \in \mathbb{R}. \tag{1.5}$$

This boundary triplet differs from that used in [13, Section 9]. It is more suitable for the investigation of the *ac*-spectrum of realizations of $\mathcal{A}$ than that one of [13, Section 9]. Due to the property (1.5) the statement (iv) of Theorem 1.2 follows immediately from our recent result [38, Theorem 1.1]. We note that this is more than one can expect when applying the classical Kato–Rosenblum theorem [25,41]. Indeed, in accordance with its generalization by Kuroda [30,31], Birman [5] and Birman and Krein [8] it is required that the resolvent differences in Theorem 1.2(iv), (v), belong to the trace class ideal $\mathfrak{S}_1$ in place of the ideal $\mathfrak{S}_\infty$ of compact operators.

The paper is organized as follows. In Section 2 we present a short introduction into the theory of boundary triplets and the corresponding Weyl functions. We recall here some statements on spectral multiplicities as well as the main result from [38] that are used in the sequel.

In Section 3 we obtain some new results on symmetric operators $S := \bigoplus_{n=1}^\infty S_n$ being an infinite direct sum of closed symmetric operators S_n with equal deficiency indices. First, let $\Pi_n = \{\mathcal{H}_n, \Gamma_{0n}, \Gamma_{1n}\}$ be a boundary triplet for S_n^*, $n \in \mathbb{N}$. In general, the direct sum $\Pi = \bigoplus_{n=1}^\infty \Pi_n$ is not a boundary triplet for $S^* = \bigoplus_{n=1}^\infty S_n^*$, cf. [26]. Nevertheless, we show, cf. Theorem 3.7, that each boundary triplet Π_n can slightly be modified such that the new sequence $\widetilde{\Pi}_n = \{\mathcal{H}_n, \widetilde{\Gamma}_{0n}, \widetilde{\Gamma}_{1n}\}$ of boundary triplets possess the following properties:

(i) the direct sum

$$\widetilde{\Pi} = \bigoplus_{n=1}^\infty \widetilde{\Pi}_n = \{\mathcal{H}, \widetilde{\Gamma}_0, \widetilde{\Gamma}_1\}, \quad \mathcal{H} := \bigoplus_{n=1}^\infty \mathcal{H}_n,\ \widetilde{\Gamma}_j := \bigoplus_{n=1}^\infty \widetilde{\Gamma}_{jn},\ j \in \{0,1\},$$

is already a boundary triplet for S^*;

(ii) the extension $\widetilde{S}_0 := S^* \upharpoonright \ker \widetilde{\Gamma}_0$ satisfies $\widetilde{S}_0 = \bigoplus_{n=1}^\infty \widetilde{S}_{0n}$ where

$$\widetilde{S}_{0n} := S_n^* \upharpoonright \ker \widetilde{\Gamma}_{0n} = S_n^* \upharpoonright \ker \Gamma_{0n} =: S_{0n}, \quad n \in \mathbb{N}.$$

Moreover, the Weyl function $\widetilde{M}(\cdot)$ corresponding to the triplet $\widetilde{\Pi}$ is block-diagonal, that is, $\widetilde{M}(\cdot) = \bigoplus_{n=1}^\infty \widetilde{M}_n(\cdot)$ where $\widetilde{M}_n(\cdot)$ is the Weyl function corresponding to the triplet $\widetilde{\Pi}_n$, $n \in \mathbb{N}$. This result plays a crucial role in the sequel. In particular, we show that the self-adjoint extension $S_0 = \bigoplus_{n=1}^\infty S_{0n}$ is *ac*-minimal provided that the deficiency indices $n_\pm(S_n)$ are equal and finite. We also prove in this section that if $S_n \geqslant 0$, $n \in \mathbb{N}$, then Friedrichs and Krein extensions S^F and S^K of $S := \bigoplus_{n=1}^\infty S_n$ are direct sums of Friedrichs and Krein extensions of the summands S_n, respectively, i.e., $S^F := \bigoplus_{n=1}^\infty S_n^F$ and $S^K := \bigoplus_{n=1}^\infty S_n^K$, cf. Corollary 3.10. In a recent paper [27] Theorem 3.7 has been applied to 1*D* Schrödinger operators with local point interactions.

In Section 4 we consider Sturm–Liouville operators with bounded operator potentials. In this case it is easy to construct a boundary triplet for A^*. We prove here Theorem 1.2 in the case $T \in [\mathcal{H}]$ and establish some additional properties of Krein's realization as well as other realizations.

In Section 5 we extend the results of Section 4 to the case of Sturm–Liouville operators with unbounded non-negative operator potentials. We construct here a boundary triplet for A^* using results of both Sections 3 and 4 and compute the (block-diagonal) Weyl function. Based on this construction we prove Theorem 1.2 for unbounded T and establish some additional properties of Dirichlet, Neumann and other realizations as well. In particular, we prove here the regularity results mentioned above. Finally, in Section 6 we apply the abstract results to the elliptic partial differential expression $\mathcal{L}$ in the half-space.

Appendix A is devoted to realizations of $\mathcal{A}$ admitting separation of variables, that is, to realization having a special tensor structure $A_\tau = l_\tau \otimes I_\mathcal{H} + I_\mathcal{K} \otimes T$ where l_τ is the Sturm–Liouville operator in $L^2(\mathbb{R}_+)$. The investigation of spectral properties of these realizations is much simpler. However, *only such type realizations of $\mathcal{A}$ can be treated in this way*. For the reader's convenience Appendices B, C, D are added. Here we briefly describe the multiplicity function of non-orthogonal operator-valued measures, recall the definition of *ac*-closure and its simple properties and finally present basic facts on linear relations.

The main results of the paper have been announced (without proofs) in [37], a preliminary version has been published as a preprint [36].

Notations In the following we consider only separable Hilbert spaces which are denoted by $\mathfrak{H}$, $\mathcal{H}$, etc. A closed linear relation in $\mathcal{H}$ is a closed subspace of $\mathcal{H} \oplus \mathcal{H}$. The set of all closed linear relations in $\mathcal{H}$ is denoted by $\widetilde{\mathcal{C}}(\mathcal{H})$. A graph $\mathrm{gr}(B)$ of a closed linear operator B belongs to $\widetilde{\mathcal{C}}(\mathcal{H})$. The symbols $\mathcal{C}(\mathcal{H}_1, \mathcal{H}_2)$ and $[\mathfrak{H}_1, \mathfrak{H}_2]$ stand for the sets of closed and bounded linear operators from $\mathfrak{H}_1$ to $\mathfrak{H}_2$, respectively. We set $\mathcal{C}(\mathcal{H}) := \mathcal{C}(\mathcal{H}, \mathcal{H})$ and $[\mathfrak{H}] := [\mathfrak{H}, \mathfrak{H}]$. We regard $\mathcal{C}(\mathcal{H})$ as a subset of $\widetilde{\mathcal{C}}(\mathcal{H})$ identifying an operator B with its graph $\mathrm{gr}(B)$.

The Schatten–von Neumann ideals of compact operators are denoted by $\mathfrak{S}_p(\mathfrak{H})$, $p \in [1, \infty]$. In particular, $\mathfrak{S}_1(\mathfrak{H})$, $\mathfrak{S}_2(\mathfrak{H})$ and $\mathfrak{S}_\infty(\mathfrak{H})$ stand for the trace class, Hilbert–Schmidt operators and compact operators, respectively.

The symbols $\mathrm{dom}(T)$, $\mathrm{ran}(T)$, $\varrho(T)$ and $\sigma(T)$ stand for the domain, the range, the resolvent set and the spectrum of an operator $T \in \mathcal{C}(\mathcal{H})$, respectively; T^{ac} and $\sigma_{ac}(T)$ denote the absolutely continuous part and the absolutely continuous spectrum of a self-adjoint operator $T = T^*$.

2. Preliminaries

2.1. Boundary triplets and proper extensions

In this section we briefly recall basic facts on boundary triplets and the corresponding Weyl functions, cf. [12–14,20].

Let A be a densely defined closed symmetric operator in a separable Hilbert space $\mathfrak{H}$ with equal deficiency indices $n_\pm(A) = \dim(\ker(A^* \mp i)) \leqslant \infty$.

Definition 2.1. (See [20].) A totality $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$, where $\mathcal{H}$ is an auxiliary Hilbert space and $\Gamma_0, \Gamma_1 : \mathrm{dom}(A^*) \to \mathcal{H}$ are linear mappings (trace operators), is called a boundary triplet for A^* if the "second abstract Green's identity"

$$\bigl(A^* f, g\bigr) - \bigl(f, A^* g\bigr) = (\Gamma_1 f, \Gamma_0 g)_\mathcal{H} - (\Gamma_0 f, \Gamma_1 g)_\mathcal{H}, \quad f, g \in \mathrm{dom}\bigl(A^*\bigr), \tag{2.1}$$

holds and the mapping $\Gamma := (\Gamma_0, \Gamma_1)^\top : \mathrm{dom}(A^*) \to (\mathcal{H} \oplus \mathcal{H})^\top$ is surjective.

Let us agree in what follows that *the notation $A \subset A'$ means only the strict inclusion*, i.e. the equality $A = A'$ is excluded.

Definition 2.2. (See [20].) (i) A closed extension A' of A is called a *proper extension*, if $A \subset A' \subset A^*$. The set of all proper extensions of A completed by the (non-proper) extensions A and A^* is denoted by Ext_A.

(ii) Two proper extensions A', A'' are called disjoint if $\mathrm{dom}(A') \cap \mathrm{dom}(A'') = \mathrm{dom}(A)$ and transversal if in addition $\mathrm{dom}(A') + \mathrm{dom}(A'') = \mathrm{dom}(A^*)$.

Any self-adjoint extension $\widetilde{A} = \widetilde{A}^*$ is proper, $\widetilde{A} \in \mathrm{Ext}_A$. Moreover, if A' and A'' are disjoint and self-adjoint, then $\mathrm{dom}(A') + \mathrm{dom}(A'')$ is dense in $\mathrm{dom}(A^*)$.

A boundary triplet $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ for A^* exists whenever $n_+(A) = n_-(A)$. Moreover, the relations $n_\pm(A) = \dim(\mathcal{H})$ and $\ker(\Gamma_0) \cap \ker(\Gamma_1) = \mathrm{dom}(A)$ are valid. In addition, one has $\Gamma_0, \Gamma_1 \in [\mathfrak{H}_+, \mathcal{H}]$ where the Hilbert space $\mathfrak{H}_+$ denotes $\mathrm{dom}(A^*)$ equipped with the graph norm of A^*.

The concept of boundary triplets makes it possible to parameterize the set Ext_A in terms of abstract boundary conditions. To this end we denote by $\widetilde{\mathcal{C}}(\mathcal{H})$ the set of closed linear relations in $\mathcal{H}$. For basic definitions related to linear relations we refer to Appendix D. In particular, the definitions of the inverse and of the resolvent set of a linear relation are given there.

Proposition 2.3. *Let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a boundary triplet for A^*. Then the mapping*

$$\mathrm{Ext}_A \ni \widetilde{A} \to \Gamma\,\mathrm{dom}(\widetilde{A}) = \bigl\{\{\Gamma_0 f, \Gamma_1 f\}\colon\ f \in \mathrm{dom}(\widetilde{A})\bigr\} =: \Theta \in \widetilde{\mathcal{C}}(\mathcal{H}) \tag{2.2}$$

establishes a bijective correspondence between the sets Ext_A *and* $\widetilde{\mathcal{C}}(\mathcal{H})$. *We put $A_\Theta := \widetilde{A}$ where Θ is defined by* (2.2)*. Moreover, the following hold*:

(i) *A_Θ is symmetric* (*self-adjoint*) *if and only if Θ is symmetric* (*self-adjoint*)*.*
(ii) *The extensions A_Θ and A_0 are disjoint if and only if there exists a closed operator B in $\mathcal{H}$ such that* $\mathrm{gr}(B) = \Theta$. *In this case* (2.2) *takes the form*

$$A_\Theta = A^* \restriction \ker(\Gamma_1 - B\Gamma_0) =: A_B. \tag{2.3}$$

(iii) *The extensions A_B and A_0 are transversal if and only if $B \in [\mathcal{H}]$.*

The linear relation Θ (the operator B) in the correspondence (2.2) (respectively (2.3)) is called *the boundary relation* (*the boundary operator*).

We emphasize that *in the case of differential operators opposed to the von Neumann parameterization the parameterization* (2.2)–(2.3) *describes the set of proper extensions directly in terms of boundary conditions.*

With any boundary triplet Π one associates two special extensions $A_j := A^* \restriction \ker(\Gamma_j)$, $j \in \{0,1\}$, which are self-adjoint. Indeed, $A_j = A_{\Theta_j}$, $j \in \{0,1\}$, where $\Theta_0 := \{0\} \times \mathcal{H}$ and $\Theta_1 := \mathcal{H} \times \{0\}$. By Proposition 2.3, $A_j = A_j^*$ since $\Theta_j = \Theta_j^*$. In the sequel the extension A_0 is usually regarded as a reference self-adjoint extension. Further, if Θ is the graph of a closed operator B, i.e. $\Theta = \mathrm{gr}(B)$, then the operator A_Θ is denoted by A_B.

Note also that for any extension $A_0 = A_0^* \in \mathrm{Ext}_A$ there exists a boundary triplet $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ for A^* such that $A_0 := A^* \restriction \ker(\Gamma_0)$ (see [13]).

2.2. Weyl functions and γ-fields

It is well known that Weyl functions are an important tool in the direct and inverse spectral theory of singular Sturm–Liouville operators. In [13,14,12] the concept of Weyl function was generalized to the case of an arbitrary symmetric operator A with $n_+(A) = n_-(A)$. Following [13,14,12] we recall basic facts on Weyl functions and γ-fields associated with a boundary triplet Π.

Definition 2.4. (See [13,12].) Let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a boundary triplet for A^*. The functions $\gamma(\cdot)\colon \varrho(A_0) \to [\mathcal{H}, \mathfrak{H}]$ and $M(\cdot)\colon \varrho(A_0) \to [\mathcal{H}]$ defined by

$$\gamma(z) := (\Gamma_0 \upharpoonright \mathfrak{N}_z)^{-1} \quad \text{and} \quad M(z) := \Gamma_1 \gamma(z), \quad z \in \varrho(A_0), \tag{2.4}$$

are called the γ*-field* and the *Weyl function*, respectively, corresponding to Π.

It follows from the identity $\mathrm{dom}(A^*) = \ker(\Gamma_0) \dotplus \mathfrak{N}_z$, $z \in \varrho(A_0)$, where $A_0 = A^* \upharpoonright \ker(\Gamma_0)$, and $\mathfrak{N}_z := \mathfrak{N}_z(A) := \ker(A^* - z)$, that the γ-field $\gamma(\cdot)$ is well defined and takes values in $[\mathcal{H}, \mathfrak{H}]$. Since $\Gamma_1 \in [\mathfrak{H}_+, \mathcal{H}]$, it follows from (2.4) that $M(\cdot)$ is well defined too and takes values in $[\mathcal{H}]$. Moreover, both $\gamma(\cdot)$ and $M(\cdot)$ are holomorphic on $\varrho(A_0)$. It turns out that the Weyl function $M(\cdot)$ is in fact an $R_{\mathcal{H}}$-function (Nevanlinna or Herglotz function), that is, $M(\cdot)$ is an $[\mathcal{H}]$-valued holomorphic function on $\mathbb{C} \setminus \mathbb{R}$ satisfying

$$M(z) = M(\bar{z})^* \quad \text{and} \quad \frac{\mathrm{Im}(M(z))}{\mathrm{Im}(z)} \geqslant 0, \quad z \in \mathbb{C} \setminus \mathbb{R},$$

which in addition satisfies the condition $0 \in \varrho(\mathrm{Im}(M(z)))$, $z \in \mathbb{C} \setminus \mathbb{R}$.

Recall that a symmetric operator is said to be *simple* if there is no non-trivial subspace which reduces it to a self-adjoint operator. It is easily seen (and well known) that A is simple if and only if $\mathrm{span}\{\mathfrak{N}_z(A){:}\ z \in \mathbb{C} \setminus \mathbb{R}\} = \mathfrak{H}$.

If A is simple, then the Weyl function $M(\cdot)$ determines the pair $\{A, A_0\}$ as well as the boundary triplet uniquely up to unitary equivalence (see [13,14,29]). Thus, $M(\cdot)$ contains (implicitly) full information on spectral properties of A_0.

For a fixed extension $A_0 = A_0^*$ of A a boundary triplet $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ satisfying $\mathrm{dom}(A_0) = \ker(\Gamma_0)$ is not unique. We describe the situation in the following lemma.

Lemma 2.5. *Let* $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ *and* $\widetilde{\Pi} = \{\widetilde{\mathcal{H}}, \widetilde{\Gamma}_0, \widetilde{\Gamma}_1\}$ *be boundary triplets for* A^* *satisfying* $\ker(\Gamma_0) = \ker(\widetilde{\Gamma}_0)$ *and* $\widetilde{A} \in \mathrm{Ext}_A$. *Let also* $M(\cdot)$, $\widetilde{M}(\cdot)$ *and* $\gamma(\cdot)$, $\widetilde{\gamma}(\cdot)$ *be the corresponding Weyl functions and the* γ*-fields, respectively and* $\Theta = \Gamma\,\mathrm{dom}(\widetilde{A})$, $\widetilde{\Theta} = \widetilde{\Gamma}\,\mathrm{dom}(\widetilde{A})$. *Then the following hold*:

(i) *There exist operators* $R_0 = R_0^* \in [\widetilde{\mathcal{H}}]$ *and* $R \in [\widetilde{\mathcal{H}}, \mathcal{H}]$ *with bounded inverse* R^{-1} *such that*

$$\Gamma_0 = R\widetilde{\Gamma}_0 \quad \text{and} \quad \Gamma_1 = \left(R^{-1}\right)^* (\widetilde{\Gamma}_1 - R_0 \widetilde{\Gamma}_0).$$

(ii) *The following identities hold*

$$\widetilde{\gamma}(z) = \gamma(z) R, \qquad \widetilde{M}(z) = R^* M(z) R + R_0, \qquad \widetilde{\Theta} = R^* \Theta R + R_0. \tag{2.5}$$

2.3. Krein type formula for resolvents and resolvent comparability

With any boundary triplet $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ for A^* it is associated the following Krein type formula (cf. [13,14,12])

$$(A_\Theta - z)^{-1} - (A_0 - z)^{-1} = \gamma(z)\bigl(\Theta - M(z)\bigr)^{-1} \gamma(\bar{z})^*, \quad z \in \varrho(A_0) \cap \varrho(A_\Theta). \tag{2.6}$$

It is established a one-to-one correspondence between the proper extensions $\widetilde{A} = A_\Theta$ with non-empty resolvent set and the corresponding linear relations Θ.

Formula (2.6) is a generalization of the known Krein formula for resolvents. We note also, that all objects in (2.6) are expressed by means of the trace mappings Γ_0, Γ_1 (see formulas (2.4) and (2.3)) (cf. [12–14]).

Remark 2.6. Let us recall briefly Krein's description of self-adjoint extensions of A (see, for instance, [28,29]). Following Krein one fixes $A_0 = A_0^*$ and defines a (non-unique) γ-field $\gamma(\cdot)$. Then a Q-function $Q(\cdot)$ is defined by the equality $Q(z) - Q(\overline{\zeta}) = (z - \overline{\zeta})\gamma(\zeta)^*\gamma(z)$. Clearly, it is defined uniquely up to an additive self-adjoint constant. Then Krein's formula

$$(\widetilde{A}-z)^{-1}-(A_0-z)^{-1}=\gamma(z)\bigl(\Theta-Q(z)\bigr)^{-1}\gamma(\bar{z})^*,\quad z\in\varrho(A_0)\cap\varrho(A_\Theta)\tag{2.7}$$

establishes a bijective correspondence between the set of extensions $\widetilde{A}=\widetilde{A}^*\in \mathrm{Ext}_A$ and the set of self-adjoint relations $\Theta=\Theta^*$ in $\mathcal{H}$.

Lemma 2.5 explains non-uniqueness arising in Krein's approach: no boundary triplet is fixed. However, fixing a boundary triplet $\Pi=\{\mathcal{H},\Gamma_0,\Gamma_1\}$ such that $A_0=A^*\restriction\ker(\Gamma_0)$, we eliminate the non-uniqueness in Krein's parameterization (2.7) since now γ-field $\gamma(\cdot)$ and Weyl function $M(\cdot)$ are defined uniquely by (2.4). We emphasize that namely *the two parameterizations* (2.2)–(2.3) *and* (2.6) *of* Ext_A *make it possible to apply Krein's formula to boundary value problems.*

The following result is deduced from formula (2.6) (cf. [13, Theorem 2]).

Proposition 2.7. *Let $\Pi=\{\mathcal{H},\Gamma_0,\Gamma_1\}$ be a boundary triplet for A^*, $\Theta_i=\Theta_i^*\in\widetilde{\mathcal{C}}(\mathcal{H})$, $i\in\{1,2\}$. Then for any Schatten–von Neumann ideal $\mathfrak{S}_p$, $p\in(0,\infty]$, and any $z\in\mathbb{C}\setminus\mathbb{R}$ the following equivalence holds*

$$(A_{\Theta_1}-z)^{-1}-(A_{\Theta_2}-z)^{-1}\in\mathfrak{S}_p(\mathfrak{H})\quad\Longleftrightarrow\quad(\Theta_1-z)^{-1}-(\Theta_2-z)^{-1}\in\mathfrak{S}_p(\mathcal{H}).$$

In particular, $(A_{\Theta_1}-z)^{-1}-(A_0-z)^{-1}\in\mathfrak{S}_p(\mathfrak{H})\Longleftrightarrow(\Theta_1-i)^{-1}\in\mathfrak{S}_p(\mathcal{H})$.

If in addition $\Theta_1,\Theta_2\in[\mathcal{H}]$, then for any $p\in(0,\infty]$ the equivalence holds

$$(A_{\Theta_1}-z)^{-1}-(A_{\Theta_2}-z)^{-1}\in\mathfrak{S}_p(\mathfrak{H})\quad\Longleftrightarrow\quad\Theta_1-\Theta_2\in\mathfrak{S}_p(\mathcal{H}).$$

2.4. Spectral multiplicity function and unitary equivalence

Let as above A be a densely defined *simple* closed symmetric operator in $\mathfrak{H}$ and let $\Pi=\{\mathcal{H},\Gamma_0,\Gamma_1\}$ be a boundary triplet for A^*, $M(\cdot)$ the corresponding Weyl function $M(\cdot)$ and $A_0=A^*\restriction\ker(\Gamma_0)=A_0^*$.

In our recent publication [38] using some results from [35] we expressed the spectral multiplicity function $N_{A_0^{ac}}(\cdot)$ of A_0^{ac} by means of the limit values of the Weyl function $M(\cdot)$. In general, the limit $M(t):=\text{s-lim}_{y\downarrow 0}M(t+iy)$, $t\in\mathbb{R}$, does not exist. However, for any $D\in\mathfrak{S}_2(\mathcal{H})$ satisfying $\ker(D)=\ker(D^*)=\{0\}$ the "sandwiched" Weyl function,

$$M^D(z):=D^*M(z)D,\quad z\in\mathbb{C}_\pm,$$

admits limit values $M^D(t):=\text{s-lim}_{y\downarrow 0}M^D(t+iy)$ for a.e. $t\in\mathbb{R}$, even in $\mathfrak{S}_2$-norm (cf. [6,17]). We set

$$d_{M^D}(t):=\dim\bigl(\mathrm{ran}\bigl(\mathrm{Im}\bigl(M^D(t)\bigr)\bigr)\bigr)\quad\text{for a.e. } t\in\mathbb{R}.$$

The function $d_{M^D}(\cdot)$ is Lebesgue measurable and takes values in the set of extended natural numbers $\{0\}\cup\mathbb{N}\cup\{\infty\}=\{0,1,2,\dots,\infty\}$. The set $\mathrm{supp}_{d_{M^D}}:=\{t\in\mathbb{R}:\ d_{M^D}(t)>0\}$ is called the support of $d_{M^D}(\cdot)$ and is, of course, a Lebesgue measurable set of $\mathbb{R}$. If the limit $M(t):=\text{s-lim}_{y\downarrow 0}M(t+iy)$ exists for a.e. $t\in\mathbb{R}$, then we set $d_M(t):=\dim(\mathrm{ran}(\mathrm{Im}(M(t))))$.

To state the next result we introduce the notion of the ac-closure $\mathrm{cl}_{ac}(\delta)$ of a Borel subset $\delta\subset\mathbb{R}$, see Appendix C as well as [38, Appendix] or [10,15]. An application of this notion to the investigation of the *ac*-spectrum of Schrödinger and other operators can be found in the recent publication [16].

Proposition 2.8. *(See [38, Proposition 3.2].) Let A be as above and let $\Pi=\{\mathcal{H},\Gamma_0,\Gamma_1\}$ be a boundary triplet for A^*, $M(\cdot)$ the corresponding Weyl function. If D is a Hilbert–Schmidt operator such that $\ker(D)=\ker(D^*)=\{0\}$, then $N_{A_0^{ac}}(t)=d_{M^D}(t)$ for a.e. $t\in\mathbb{R}$ and $\sigma_{ac}(A_0)=\mathrm{cl}_{ac}(\mathrm{supp}(d_{M^D}))$.*

If, in addition, the limit $M(t):=\text{s-lim}_{y\downarrow 0}M(t+iy)$ exists for a.e. $t\in\mathbb{R}$, then $N_{A_0^{ac}}(t)=d_M(t)$ for a.e. $t\in\mathbb{R}$ and $\sigma_{ac}(A_0)=\mathrm{cl}_{ac}(\mathrm{supp}(d_M))$.

If $\widetilde{A} = \widetilde{A}^* \in \mathrm{Ext}_A$ and is disjoint with A_0, then by Proposition 2.3(ii), there exists an operator $B = B^* \in \mathcal{C}(\mathcal{H})$ such that $\widetilde{A} = A_B := A^* \upharpoonright \ker(\Gamma_1 - B\Gamma_0)$. In this case the multiplicity function $N_{A_B^{ac}}(\cdot)$ is expressed by means of the generalized Weyl function $M_B(\cdot)$ of $\widetilde{A} = A_B$ defined by

$$M_B(z) := \big(B - M(z)\big)^{-1}, \quad z \in \mathbb{C}_\pm. \tag{2.8}$$

We note that if $B = B^* \in [\mathcal{H}]$, then $M_B(\cdot)$ is the Weyl function of the boundary triplet $\Pi^B = \{\mathcal{H}, \Gamma_0^B, \Gamma_1^B\} := \{\mathcal{H}, \Gamma_1 - B\Gamma_0, -\Gamma_0\}$ in the sense of Definition 2.4. However, if $B \in \mathcal{C}(\mathcal{H}) \setminus [\mathcal{H}]$, then the triplet $\Pi^B = \{\mathcal{H}, \Gamma_0^B, \Gamma_1^B\}$ is only a generalized boundary triplet for A^* in the sense of [14, Section 5] and $M_B(\cdot)$ is the corresponding Weyl function of Π^B. Therefore, loosely speaking, $M_B(\cdot)$ is called the generalized Weyl function of the extension $\widetilde{A} = A_B$ (see [2,38]).

Corollary 2.9. *(See [38, Corollary 3.3].) Let A, Π, $M(\cdot)$ and D be as in Proposition* 2.8 *and let $B = B^* \in \mathcal{C}(\mathcal{H})$. Then $N_{A_B^{ac}}(t) = d_{M_B^D}(t)$ for a.e. $t \in \mathbb{R}$ and $\sigma_{ac}(A_B) = \mathrm{cl}_{ac}(\mathrm{supp}(d_{M_B^D}))$.*

If, in addition, the limit $M_B(t) := \text{s-}\lim_{y\downarrow 0} M_B(t+iy)$ exists for a.e. $t \in \mathbb{R}$, then $N_{A_B^{ac}}(t) = d_{M_B}(t)$ for a.e. $t \in \mathbb{R}$ and $\sigma_{ac}(A_B) = \mathrm{cl}_{ac}(\mathrm{supp}(d_{M_B}))$.

Finally, we retranslate the unitary equivalence of *ac*-parts of two self-adjoint extensions in terms of limit values of Weyl functions.

Proposition 2.10. *(See [38, Theorem 3.4].) Assume the conditions of Proposition* 2.8 *and let $B = B^* \in \mathcal{C}(\mathcal{H})$. Let also $E_{A_B}(\cdot)$ and $E_{A_0}(\cdot)$ be the spectral measures of $A_B = A_B^*$ and A_0, respectively. If δ is a Borel subset of $\mathbb{R}$, then*

(i) *$A_0 E_{A_0}^{ac}(\delta)$ is unitarily equivalent to a part of $A_B E_{A_B}^{ac}(\delta)$ if and only if $d_{M^D}(t) \leqslant d_{M_B^D}(t)$ for a.e. $t \in \delta$;*
(ii) *$A_0 E_{A_0}^{ac}(\delta)$ and $A_B E_{A_B}^{ac}(\delta)$ are unitarily equivalent if and only if $d_{M^D}(t) = d_{M_B^D}(t)$ for a.e. $t \in \delta$.*

Proposition 2.10 reduces the problem of unitary equivalence of *ac*-parts of certain self-adjoint extensions of A to the computation of the functions $d_{M^D}(\cdot)$ and $d_{M_B^D}(\cdot)$. If $\delta = \mathbb{R}$, then the *ac*-part A_0^{ac} is unitarily equivalent to $\widetilde{A}^{ac} = A_B^{ac}$ if and only if $d_{M^D}(t) = d_{M_B^D}(t)$ for a.e. $t \in \mathbb{R}$.

If $M(\cdot)$ is the Weyl function of a boundary triplet Π, then we introduce the maximal normal function

$$m^+(t) := \sup_{y\in(0,1]} \big\|M(t+iy)\big\|, \quad t \in \mathbb{R}.$$

Theorem 2.11. *(See [38, Theorem 4.3, Corollary 4.6].) Assume the conditions of Proposition* 2.8*. Let $\widetilde{A} = \widetilde{A}^* \in \mathrm{Ext}_A$ and $A_0 := A^* \upharpoonright \ker(\Gamma_0)$. Assume also that there is a Borel subset δ of $\mathbb{R}$ such that the maximal normal function $m^+(t)$ is finite for a.e. $t \in \delta$ and*

$$(\widetilde{A} - i)^{-1} - (A_0 - i)^{-1} \in \mathfrak{S}_\infty(\mathfrak{H}). \tag{2.9}$$

Then the ac-parts $\widetilde{A}^{ac} E_{\widetilde{A}}(\delta)$ and $A_0^{ac} E_{A_0}(\delta)$ of $\widetilde{A} E_{\widetilde{A}}(\delta)$ and $A_0 E_{A_0}(\delta)$, respectively, are unitarily equivalent. In particular, if $m^+(t)$ is finite for a.e. $t \in \mathbb{R}$, then the operators $\widetilde{A}^{ac}$ and A_0^{ac} are unitarily equivalent.

In fact, we proved in [38, Remark 4.4] the following "individual" result.

Theorem 2.12. *Assume the conditions of Proposition* 2.8*. Assume also that $B = B^* \in \mathcal{C}(\mathcal{H})$ has discrete spectrum and $0 \in \rho(B)$. If the weak limit*

$$\text{w-}\lim_{y\downarrow 0} M^D(t+iy) = \text{w-}\lim_{y\downarrow 0} |B|^{-1/2} M(t+iy) |B|^{-1/2} =: M^D(t+i0) \tag{2.10}$$

exists for a.e. $t \in \mathbb{R}$, *where* $D := |B|^{-1/2}$, *then the ac-parts* A_B^{ac} *and* A_0^{ac} *of* A_B *and* A_0, *respectively, are unitarily equivalent.*

Notice that due to the assumption of Theorem 2.12 $B^{-1} \in \mathfrak{S}_\infty$. Therefore condition (2.9) follows from Proposition 2.7.

Note also that Theorem 2.12 generalizes the Kato–Rosenblum theorem since condition (2.10) is satisfied whenever $B^{-1} \in \mathfrak{S}_1$.

Remark 2.13. Notice that condition (1.5) is not necessary for the unitary equivalence of operators $\widetilde{A}^{ac}$ and A_0^{ac} under the condition (2.9). A counterexample gives the Neumann realization A^N of $\mathcal{A}$. Indeed, in this case the conclusion of Theorem 2.11 remains valid though the limit (1.5) of the corresponding Weyl function does not exist for every $t \in \mathbb{R}$ (cf. Theorem 1.2(v)) (see Remark 4.8).

One easily verifies that $m^+(t) < \infty$ for a.e. $t \in \delta$ if and only if limit (1.5) exists for a.e. $t \in \delta$, cf. [38, Theorem 1.2]. In turn, if condition (2.10) holds for all $B^{-1} \in \mathfrak{S}_\infty(\mathcal{H})$, then it is equivalent to condition (1.5).

However, the function $m^+(\cdot)$ depends on the choice of the boundary triplet. In [36,38] we introduced the invariant maximal normal function $\mathfrak{m}^+(\cdot)$ defined by

$$\mathfrak{m}^+(t) := \sup_{y \in (0,1]} \left\| \frac{1}{\sqrt{\operatorname{Im}(M(i))}} \big(M(t+iy) - \operatorname{Re}\big(M(i)\big)\big) \frac{1}{\sqrt{\operatorname{Im}(M(i))}} \right\|, \tag{2.11}$$

$t \in \mathbb{R}$. It follows from (2.5) that the invariant maximal normal functions for two boundary triplets $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ and $\widetilde{\Pi} = \{\widetilde{H}, \widetilde{\Gamma}_0, \widetilde{\Gamma}_1\}$ for A^* coincide whenever $A^* \upharpoonright \ker(\Gamma_0) = A^* \upharpoonright \ker(\widetilde{\Gamma}_0)$. Clearly, $\mathfrak{m}^+(t) < \infty$ if and only if $m^+(t) < \infty$ for any $t \in \mathbb{R}$. However, the invariant maximal normal function is more convenient in applications. We demonstrate this fact in the next section applying this concept to infinite direct sums of symmetric operators.

3. Direct sums of symmetric operators

3.1. Definitions and examples

Let S_n be a closed densely defined symmetric operators in $\mathfrak{H}_n$, $n_+(S_n) = n_-(S_n)$, and let $\Pi_n = \{\mathcal{H}_n, \Gamma_{0n}, \Gamma_{1n}\}$ be a boundary triplet for S_n^*, $n \in \mathbb{N}$. Let

$$A := \bigoplus_{n=1}^\infty S_n, \qquad \operatorname{dom}(A) := \bigoplus_{n=1}^\infty \operatorname{dom}(S_n). \tag{3.1}$$

Clearly, A is a closed densely defined symmetric operator in the Hilbert space $\mathfrak{H} := \bigoplus_{n=1}^\infty \mathfrak{H}_n$ with $n_\pm(A) = \infty$. Obviously, we have

$$A^* = \bigoplus_{n=1}^\infty S_n^*, \qquad \operatorname{dom}\big(A^*\big) = \bigoplus_{n=1}^\infty \operatorname{dom}\big(S_n^*\big). \tag{3.2}$$

Let us consider the direct sum $\Pi := \bigoplus_{n=1}^\infty \Pi_n =: \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ of boundary triplets defined by

$$\mathcal{H} := \bigoplus_{n=1}^\infty \mathcal{H}_n, \qquad \Gamma_0 := \bigoplus_{n=1}^\infty \Gamma_{0n} \quad \text{and} \quad \Gamma_1 := \bigoplus_{n=1}^\infty \Gamma_{1n}. \tag{3.3}$$

Setting as usual $A_0 := A^* \upharpoonright \ker(\Gamma_0)$ and $S_{0n} := S_n^* \upharpoonright \ker(\Gamma_{0n})$, one gets

$$A_0 := \bigoplus_{n=1}^{\infty} S_{0n}, \qquad \mathrm{dom}(A_0) := \bigoplus_{n=1}^{\infty} \mathrm{dom}(S_{0n}). \tag{3.4}$$

We note that the Green's identity

$$\big(S_n^* f_n, g_n\big) - \big(f_n, S_n^* g_n\big) = (\Gamma_{1n} f_n, \Gamma_{0n} g_n)_{\mathcal{H}_n} - (\Gamma_{0n} f_n, \Gamma_{1n} g_n)_{\mathcal{H}_n},$$

$f_n, g_n \in \mathrm{dom}(S_n^*)$, holds for every S_n^*, $n \in \mathbb{N}$. This yields that the Green's identity (2.1) holds for $A_* := A^* \upharpoonright \mathrm{dom}(\Gamma)$, $\mathrm{dom}(\Gamma) := \mathrm{dom}(\Gamma_0) \cap \mathrm{dom}(\Gamma_1) \subseteq \mathrm{dom}(A^*)$, that is, for $f = \bigoplus_{n=1}^{\infty} f_n, g = \bigoplus_{n=1}^{\infty} g_n \in \mathrm{dom}(\Gamma)$ we have

$$(A_* f, g) - (f, A_* g) = (\Gamma_1 f, \Gamma_0 g)_{\mathcal{H}} - (\Gamma_0 f, \Gamma_1 g)_{\mathcal{H}}, \quad f, g \in \mathrm{dom}(\Gamma), \tag{3.5}$$

where A^* and Γ_j are defined by (3.2) and (3.3), respectively. However, the Green's identity (3.5) cannot be extend to $\mathrm{dom}(A^*)$ in general, since $\mathrm{dom}(\Gamma)$ is smaller than $\mathrm{dom}(A^*)$ generically. It might even happen that Γ_j is not bounded as a mapping from $\mathrm{dom}(A^*)$ equipped with the graph norm into $\mathcal{H}$.

First counterexamples of direct sums $\Pi = \bigoplus_{n=1}^{\infty} \Pi_n$ which do not form a boundary triplet appeared in [26]. Let us indicate few simple counterexamples.

Example 3.1. Let S_n be closed symmetric operators in the Hilbert spaces $\mathfrak{H}_n$ with $n_+(S_n) = n_-(S_n)$, $n \in \mathbb{N}$, and let $A = \bigoplus_{n=1}^{\infty} S_n$ be the direct sum of S_n acting in $\mathfrak{H} := \bigoplus_{n=1}^{\infty} \mathfrak{H}_n$. Let also $\Pi_n = \{\mathcal{H}_n, \Gamma_{0n}, \Gamma_{1n}\}$ be a boundary triplet for S_n^*. Define a boundary triplet $\widetilde{\Pi}_n = \{\mathcal{H}_n, \widetilde{\Gamma}_{0n}, \widetilde{\Gamma}_{1n}\}$ for S_n^* by setting

$$\widetilde{\Gamma}_{0n} = \alpha_n \Gamma_{0n}, \qquad \widetilde{\Gamma}_{1n} = \alpha_n^{-1} \Gamma_{1n}, \qquad \alpha_n = n \cdot \|\Gamma_{0n}\|^{-1}_{[\mathfrak{H}_{+n}, \mathcal{H}_n]} \tag{3.6}$$

where $\mathfrak{H}_{+n} = \mathrm{dom}(S_n^*)$ endowed with the graph norm of S_n^*.

It is easily seen that $\widetilde{\Pi} = \bigoplus_{n=1}^{\infty} \widetilde{\Pi}_n$ is not a boundary triplet for A^*. Indeed,

$$\|\widetilde{\Gamma}_{0n}\|_{[\mathfrak{H}_{+n}, \mathcal{H}_n]} = n \|\Gamma_{0n}\|_{[\mathfrak{H}_{+n}, \mathcal{H}_n]} \cdot \|\Gamma_{0n}\|^{-1}_{[\mathfrak{H}_{+n}, \mathcal{H}_n]} = n, \quad n \in \mathbb{N}.$$

Hence $\|\widetilde{\Gamma}_0\|_{[\mathfrak{H}_+, \mathcal{H}]} = \sup_{n \in \mathbb{N}} \|\widetilde{\Gamma}_{0n}\|_{[\mathfrak{H}_{+n}, \mathcal{H}_n]} = \infty$ where $\mathfrak{H}_+ := \mathrm{dom}(A^*)$ equipped with the graph norm of A^* and $\mathcal{H}$ is given by (3.3). Thus, $\widetilde{\Gamma}_0 \notin [\mathfrak{H}_+, \mathcal{H}]$, hence, $\widetilde{\Pi}$ is not a boundary triplet for A^*. Note that

$$\mathrm{dom}(\widetilde{\Gamma}_0) := \big\{\{f_n\}_{n \in \mathbb{N}} \in \mathfrak{H}_+ \colon \{\alpha_n \Gamma_{0n} f_n\}_{n \in \mathbb{N}} \in \mathcal{H}\big\},$$
$$\mathrm{dom}(\widetilde{\Gamma}_1) := \big\{\{f_n\}_{n \in \mathbb{N}} \in \mathfrak{H}_+ \colon \big\{\alpha_n^{-1} \Gamma_{1n} f_n\big\}_{n \in \mathbb{N}} \in \mathcal{H}\big\}.$$

Consider a special case assuming that $S_n = S$ for all $n \in \mathbb{N}$. Clearly, a boundary triplet $\Pi_0 = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ for S^* is also a boundary triplet for each S_n^*, $n \in \mathbb{N}$. Setting $\Pi_n = \{\mathcal{H}_n, \Gamma_{0n}, \Gamma_{1n}\} := \Pi_0$ we easily get that $\widehat{\Pi} = \{\widehat{\mathcal{H}}, \widehat{\Gamma}_0, \widehat{\Gamma}_1\} := \bigoplus_{n=1}^{\infty} \Pi_n$ is a boundary triplet for A^*. However, if $\widetilde{\Pi}_n$ is defined by (3.6), then $\widetilde{\Pi} = \bigoplus_{n=1}^{\infty} \widetilde{\Pi}_n$ is not a boundary triplet for A^*.

Example 3.2. Let $X = \{x_n\}_1^{\infty} \subset \mathbb{R}_+$ be a sequence of positive numbers, $0 =: x_0 < x_1 < \cdots < x_n < x_{n+1} < \cdots$ and $\lim_{n \to \infty} x_n = \infty$. Let also S_n, be a symmetric operator in $\mathfrak{H}_n := L^2([x_{n-1}, x_n])$, given by

$$S_n = -i \frac{d}{dx}, \qquad \mathrm{dom}(S_n) = W_0^{1,2}\big([x_{n-1}, x_n]\big), \quad n \in \mathbb{N}. \tag{3.7}$$

The adjoint operator S_n^* is given by $S_n^* = -i\frac{d}{dx}$, $\mathrm{dom}(S_n^*) = W^{1,2}([x_{n-1}, x_n])$. Let $A := \bigoplus_{n=1}^{\infty} S_n$ and let $d_n := x_n - x_{n-1}$, $n \in \mathbb{N}$.

Further, assume that $d^*(X) := \sup_{n\in\mathbb{N}} d_n < \infty$. It can easily be shown that $f = \{f_n\}_{n\in\mathbb{N}} \in \mathrm{dom}(A^*) = \bigoplus_{n=1}^{\infty} W^{1,2}([x_{n-1}, x_n])$ if and only if the boundary values $\{f_n(x_n \pm 0)\}_{n\in\mathbb{N}}$ satisfy the following conditions

$$\sum_{n\in\mathbb{N}} \frac{|f_n(x_n - 0) - f_n(x_{n-1} + 0)|^2}{d_n} < \infty \tag{3.8}$$

and

$$\sum_{n\in\mathbb{N}} d_n\bigl(\bigl|f_n(x_{n-1} + 0)\bigr|^2 + \bigl|f_n(x_n - 0)\bigr|^2\bigr) < \infty. \tag{3.9}$$

It is easily seen that a totality $\Pi_n = \{\mathbb{C}, \Gamma_{0n}, \Gamma_{1n}\}$ with

$$\begin{aligned}\Gamma_{0n} f_n &:= i\frac{f_n(x_n - 0) - f_n(x_{n-1} + 0)}{\sqrt{2}},\\ \Gamma_{1n} f_n &:= \frac{f_n(x_n - 0) + f_n(x_{n-1} + 0)}{\sqrt{2}},\end{aligned} \tag{3.10}$$

is a boundary triplet for S_n^*, $n \in \mathbb{N}$. We set $\Pi := \bigoplus_{n=1}^{\infty} \Pi_n =: \{\mathcal{H}, \Gamma_0, \Gamma_1\}$, where $\mathcal{H} := \bigoplus_{n=1}^{\infty} \mathcal{H}_n = l^2(\mathbb{N})$ and $\Gamma_j := \bigoplus_{n=1}^{\infty} \Gamma_{jn}$, $j \in \{0, 1\}$.

It follows from (3.8) that $\mathrm{dom}(\Gamma_0) = \mathrm{dom}(A^*)$ and Γ_0 is bounded. Indeed, for any $f = \{f_n\}_{n\in\mathbb{N}} \in \bigoplus_{n=1}^{\infty} W^{1,2}([x_{n-1}, x_n]) = \mathrm{dom}(A^*)$ one gets

$$\begin{aligned}\|\Gamma_0 f\|_{\mathcal{H}}^2 &= \bigl\|\Gamma_0\{f_n\}_{n\in\mathbb{N}}\bigr\|_{\mathcal{H}}^2 = \frac{1}{2}\sum_{n\in\mathbb{N}}\bigl|f_n(x_{n-1} + 0) - f_n(x_n - 0)\bigr|^2\\ &\leqslant \frac{d^*(X)}{2}\sum_{n\in\mathbb{N}} \frac{|f_n(x_{n-1} + 0) - f_n(x_n - 0)|^2}{d_n} \leqslant \frac{d^*(X)}{2}\sum_{n\in\mathbb{N}} \|f_n\|_{W^{1,2}([x_{n-1}, x_n])}^2 < \infty.\end{aligned} \tag{3.11}$$

Further, combining (3.11) with the definition of Γ_1 we find

$$\begin{aligned}\mathrm{dom}(\Gamma_1) &= \mathrm{dom}(\Gamma_0) \cap \mathrm{dom}(\Gamma_1)\\ &= \left\{\{f_n\}_{n\in\mathbb{N}} \in \bigoplus_{n=1}^{\infty} W^{1,2}\bigl([x_{n-1}, x_n]\bigr)\colon \sum_{n\in\mathbb{N}}\bigl(\bigl|f_n(x_n - 0)\bigr|^2 + \bigl|f_n(x_{n-1} + 0)\bigr|^2\bigr) < \infty\right\}.\end{aligned} \tag{3.12}$$

If $0 < d_*(X) := \inf_{n\in\mathbb{N}} d_n$, then (3.9) yields $\mathrm{dom}(\Gamma_1) = \mathrm{dom}(A^*)$.

However, if $d_*(X) = 0$, then $\mathrm{dom}(\Gamma_1)$ is a strict part of $\mathrm{dom}(A^*)$ though it is dense in $\mathrm{dom}(A^*) = \bigoplus_{n=1}^{\infty} W^{1,2}([x_{n-1}, x_n])$. Indeed, choosing $f_n(x) := d_n^{-1/2} c_n$, $x \in [x_{n-1}, x_n]$, $c_n \in \mathbb{C}$, $n \in \mathbb{N}$, and noting that $\|f_n\|_{W^{1,2}([x_{n-1}, x_n])}^2 = |c_n|^2$ we get that $f = \{f_n\}_{n\in\mathbb{N}} \in \mathrm{dom}(A^*)$ if and only if $\{c_n\}_{n\in\mathbb{N}} \in l^2(\mathbb{N})$. Besides, by (3.12) $f \in \mathrm{dom}(\Gamma_1)$ if and only if $\sum_{n\in\mathbb{N}} d_n^{-1}|c_n|^2 < \infty$. Clearly, if $d_*(X) = 0$, the sequences satisfying the later inequality form a part of $l^2(\mathbb{N})$.

The Weyl function $M_n(\cdot)$ corresponding to the triplet Π_n is

$$M_n(z) = -i\frac{e^{izx_n} + e^{izx_{n-1}}}{e^{izx_n} - e^{izx_{n-1}}} = -\cot\bigl(2^{-1} z d_n\bigr), \quad z \in \mathbb{C}_{\pm}. \tag{3.13}$$

Clearly, $M(\cdot) = \bigoplus_{n=1}^{\infty} M_n(\cdot)$ is bounded if and only if $d_*(X) > 0$.

Setting $\widehat{\Gamma}_{0n} = d_n^{-1/2}\Gamma_{0n}$ and $\widehat{\Gamma}_{1n} = d_n^{1/2}\Gamma_{1n}$, $n \in \mathbb{N}$, we obtain a new boundary triplet $\widehat{\Pi}_n = \{\mathbb{C}, \widehat{\Gamma}_{0n}, \widehat{\Gamma}_{1n}\}$ for S_n^*. It easily follows from (3.8) and (3.9) that the direct sum $\widehat{\Pi} = \bigoplus_{n\in\mathbb{N}} \widehat{\Pi}_n$ is already a boundary triplet for A^*. Moreover, the corresponding Weyl function $\widehat{M}(\cdot)$ is given by $\widehat{M}(z) = \bigoplus_{n=1}^{\infty}(-d_n \cot(2^{-1}zd_n))$.

Remark 3.3. If $d_*(X) = 0$, then (3.11) yields that $\operatorname{ran}(\Gamma_0)$ is a proper dense part of $l^2(\mathbb{N})$. At the same time, if $d^*(X) < \infty$, then by (3.9) and (3.10), Γ_1 is surjective, $\operatorname{ran}(\Gamma_1) = l^2(\mathbb{N})$. Since, in addition, $A_1 = \bigoplus_{n=1}^{\infty} S_{n1} = A_1^*$, the triplet $\Pi^\top := \{l^2(\mathbb{N}), \Gamma_1, -\Gamma_0\}$ is a generalized boundary triplet for A^* in the sense of [14, Section 6]. Moreover, by (3.13), the corresponding Weyl function is $M_\top(\cdot) = \bigoplus_{n=1}^{\infty}(\tan(2^{-1}zd_n))$. Clearly, $M_\top(\cdot)$ is bounded for $z \in \mathbb{C}_+$. The latter correlates to the fact that $\Pi^\top$ is a generalized boundary triplet for A^*.

3.2. Boundary triplets for direct sums. Regularization construction

The previous examples show that the direct sum $\Pi = \bigoplus_{n=1}^{\infty} \Pi_n$ of boundary triplets Π_n for S_n^* is not necessarily a boundary triplet for $A^* = \bigoplus_{n=1}^{\infty} S_n^*$. This naturally leads to the following problem:

Problem 3.4. Let $\{S_n\}_{n\in\mathbb{N}}$ be a sequence of closed symmetric operators, $S_n \in \mathcal{C}(\mathfrak{H}_n)$, $n \in \mathbb{N}$, and let $\Pi_n = \{\mathcal{H}_n, \Gamma_0, \Gamma_1\}$ be a boundary triplet for S_n^*, $n \in \mathbb{N}$. Is it possible to modify the boundary triplets Π_n in such a way that the modified boundary triplets $\widetilde{\Pi}_n = \{\mathcal{H}_n, \widetilde{\Gamma}_{0n}, \widetilde{\Gamma}_{1n}\}$ for S_n^* satisfy the following conditions:

(i) $\widetilde{\Pi} = \bigoplus_{n=1}^{\infty} \widetilde{\Pi}_n$ is a boundary triplet for $A^* = \bigoplus_{n=1}^{\infty} S_n^*$;
(ii) $\widetilde{S}_{0n} := S_n^* \upharpoonright \ker(\widetilde{\Gamma}_{0n}) = S_n^* \upharpoonright \ker(\Gamma_{0n}) =: S_{0n}$, $n \in \mathbb{N}$?

If the answer to this problem is affirmative, then $\widetilde{A}_0 := \bigoplus_{n=1}^{\infty} \widetilde{S}_{0n} = \bigoplus_{n=1}^{\infty} S_{0n} = A_0$. Moreover, the Weyl function $\widetilde{M}(\cdot)$ and the γ-field $\widetilde{\gamma}(\cdot)$ corresponding to $\widetilde{\Pi}$ admit the following block-diagonal representations

$$\widetilde{M}(z) = \bigoplus_{n=1}^{\infty} \widetilde{M}_n(z) \quad \text{and} \quad \widetilde{\gamma}(z) = \bigoplus_{n=1}^{\infty} \widetilde{\gamma}_n(z), \tag{3.14}$$

where $\widetilde{M}_n(\cdot)$ and $\widetilde{\gamma}_n(\cdot)$ are the Weyl functions and the γ-field corresponding to $\widetilde{\Pi}_n$, $n \in \mathbb{N}$. Note that existence of a boundary triplet $\Pi' = \{\mathcal{H}, \Gamma_0', \Gamma_1'\}$ for A^* satisfying $\ker(\Gamma_0') = \operatorname{dom}(A_0)$ is known (see [13,20]). However, in applications we need a special boundary triplet for $A^* = \bigoplus_{n=1}^{\infty} S_n^*$ which preserves the direct sum structure and leads therefore to the block-diagonal form (3.14) of the corresponding Weyl function. Our regularization construction substantially involves the Weyl functions $M_n(\cdot)$ corresponding to Π_n. We start with a simple lemma.

Lemma 3.5. *Let S be a densely defined closed symmetric operator with equal deficiency indices, let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a boundary triplet for S^* and let $M(\cdot)$ be the corresponding Weyl function. Then there exists a boundary triplet $\widetilde{\Pi} = \{\mathcal{H}, \widetilde{\Gamma}_0, \widetilde{\Gamma}_1\}$ for S^* such that $\ker(\widetilde{\Gamma}_0) = \ker(\Gamma_0)$ and the corresponding Weyl function $\widetilde{M}(\cdot)$ satisfies $\widetilde{M}(i) = iI_{\mathcal{H}}$.*

Proof. Let $M(i) = Q + iR^2$ where $Q := \operatorname{Re}(M(i))$, $R := \sqrt{\operatorname{Im}(M(i))}$. We set

$$\widetilde{\Gamma}_0 := R\Gamma_0 \quad \text{and} \quad \widetilde{\Gamma}_1 := R^{-1}(\Gamma_1 - Q\Gamma_0). \tag{3.15}$$

A straightforward computation shows that $\widetilde{\Pi} := \{\mathcal{H}, \widetilde{\Gamma}_0, \widetilde{\Gamma}_1\}$ is a boundary triplet for A^*. Clearly, $\ker(\widetilde{\Gamma}_0) = \ker(\Gamma_0)$. The corresponding Weyl function $\widetilde{M}(\cdot)$ is given by $\widetilde{M}(\cdot) = R^{-1}(M(\cdot) - Q)R^{-1}$ which yields $\widetilde{M}(i) = iI_{\mathcal{H}}$. $\square$

If S is a densely defined closed symmetric operator in $\mathfrak{H}$, then by the first von Neumann formula the direct decomposition $\mathrm{dom}(S^*) = \mathrm{dom}(S) \dotplus \mathfrak{N}_i \dotplus \mathfrak{N}_{-i}$ holds, where $\mathfrak{N}_{\pm i} := \ker(S^* \mp i)$. Equipping $\mathrm{dom}(S^*)$ with the inner product

$$(f,g)_+ := (f,g)_{+,S^*} := \big(S^*f, S^*g\big) + (f,g), \quad f,g \in \mathrm{dom}\big(S^*\big), \tag{3.16}$$

one obtains a Hilbert space $\mathfrak{H}_+ := \mathfrak{H}_{+,S^*}$. The first von Neumann formula leads to the following orthogonal decomposition

$$\mathfrak{H}_+ := \mathfrak{H}_{+,S^*} = \mathrm{dom}(S) \oplus \mathfrak{N}_i \oplus \mathfrak{N}_{-i}.$$

Lemma 3.6. *Assume the conditions of Lemma* 3.5. *If $M(i) = iI_{\mathcal{H}}$, then $\Gamma : \mathfrak{H}_+ \to \mathcal{H} \oplus \mathcal{H}$, $\Gamma := (\Gamma_0, \Gamma_1)$ is a contraction. Moreover, Γ isometrically maps $\mathfrak{N} := \mathfrak{N}_i \oplus \mathfrak{N}_{-i}$ onto $\mathcal{H}$.*

Proof. We show that

$$\big\|\Gamma(f + f_i + f_{-i})\big\|^2_{\mathcal{H}\oplus\mathcal{H}} = \|f_i + f_{-i}\|^2_+ \tag{3.17}$$

where $f \dotplus f_i \dotplus f_{-i} \in \mathrm{dom}(S) \dotplus \mathfrak{N}_i \dotplus \mathfrak{N}_{-i} = \mathrm{dom}(S^*)$. Since $\mathrm{dom}(S) = \ker(\Gamma_0) \cap \ker(\Gamma_1)$ we find

$$\big\|\Gamma(f + f_i + f_{-i})\big\|^2_{\mathcal{H}\oplus\mathcal{H}} = \big\|\Gamma_0(f_i + f_{-i})\big\|^2_{\mathcal{H}} + \big\|\Gamma_1(f_i + f_{-i})\big\|^2_{\mathcal{H}}.$$

Clearly,

$$\big\|\Gamma_j(f_i + f_{-i})\big\|^2_{\mathcal{H}} = \|\Gamma_j f_i\|^2 + 2\,\mathrm{Re}\big((\Gamma_j f_i, \Gamma_j f_{-i})\big) + \|\Gamma_j f_{-i}\|^2, \quad j \in \{0,1\}. \tag{3.18}$$

Since $\Gamma_1 f_i = M(i)\Gamma_0 f_i = i\Gamma_0 f_i$ and $\Gamma_1 f_{-i} = M(-i)\Gamma_0 f_{-i} = -i\Gamma_0 f_{-i}$, we obtain

$$\big\|\Gamma_1(f_i + f_{-i})\big\|^2_{\mathcal{H}} = (\Gamma_0 f_i, \Gamma_0 f_i) - 2\,\mathrm{Re}\big((\Gamma_0 f_i, \Gamma_0 f_{-i})\big) + (\Gamma_0 f_{-i}, \Gamma_0 f_{-i}). \tag{3.19}$$

Taking a sum of (3.18) and (3.19) we get

$$\big\|\Gamma_0(f_i + f_{-i})\big\|^2_{\mathcal{H}} + \big\|\Gamma_1(f_i + f_{-i})\big\|^2_{\mathcal{H}} = 2\|\Gamma_0 f_i\|^2_{\mathcal{H}} + 2\|\Gamma_0 f_{-i}\|^2_{\mathcal{H}}. \tag{3.20}$$

Combining equalities $\Gamma_1 f_{\pm i} = \pm i\Gamma_0 f_{\pm i}$ with the Green's identity (2.1) we obtain $\|\Gamma_0 f_i\|_{\mathcal{H}} = \|f_i\|$ and $\|\Gamma_0 f_{-i}\|_{\mathcal{H}} = \|f_{-i}\|$. Therefore (3.20) takes the form

$$\big\|\Gamma_0(f_i + f_{-i})\big\|^2_{\mathcal{H}} + \big\|\Gamma_1(f_i + f_{-i})\big\|^2_{\mathcal{H}} = 2\|f_i\|^2 + 2\|f_{-i}\|^2. \tag{3.21}$$

A straightforward computation shows that $\|f_i + f_{-i}\|^2_+ = 2\|f_i\|^2 + 2\|f_{-i}\|^2$. Together with (3.21) this proves (3.17). Since $\|f_i + f_{-i}\|^2_+ \leqslant \|f\|^2_+ + \|f_i + f_{-i}\|^2_+ = \|f + f_i + f_{-i}\|^2_+$, we get from (3.17) that Γ is a contraction.

Obviously, Γ is an isometry from $\mathfrak{N}$ into $\mathcal{H} \oplus \mathcal{H}$. Since Π is a boundary triplet for S^*, $\mathrm{ran}(\Gamma) = \mathcal{H} \oplus \mathcal{H}$. Hence Γ is an isometry from $\mathfrak{N}$ onto $\mathcal{H} \oplus \mathcal{H}$. $\square$

Passing to the direct sum (3.1), we equip $\mathrm{dom}(S_n^*)$ and $\mathrm{dom}(A^*)$ with their graph's norms and obtain the Hilbert spaces $\mathfrak{H}_{+n}$ and $\mathfrak{H}_+$, respectively. The corresponding inner products $(f,g)_{+n}$ and $(f,g)_+$ are defined by (3.16) with S^* replaced by S_n^* and A^*, respectively. Clearly, $\mathfrak{H}_+ = \bigoplus_{n=1}^{\infty} \mathfrak{H}_{+n}$.

Now we are ready to solve Problem 3.4.

Theorem 3.7 *(Regularization theorem). Let $\{S_n\}_{n=1}^{\infty}$ be a sequence of densely defined closed symmetric operators in $\mathfrak{H}_n$ and $A := \bigoplus_{n=1}^{\infty} S_n$. Let also $\Pi_n = \{\mathcal{H}_n, \Gamma_{0n}, \Gamma_{1n}\}$ be a boundary triplet for S_n^*, $M_n(\cdot)$, $n \in \mathbb{N}$, the corresponding Weyl function, and let $Q_n := \mathrm{Re}(M_n(i))$, $R_n := \sqrt{\mathrm{Im}(M_n(i))}$. Then the sequence of triplets $\widetilde{\Pi}_n := \{\mathcal{H}_n, \widetilde{\Gamma}_{0n}, \widetilde{\Gamma}_{1n}\}$ where*

$$\widetilde{\Gamma}_{0n} := R_n \Gamma_{0n}, \qquad \widetilde{\Gamma}_{1n} := R_n^{-1}(\Gamma_{1n} - Q_n \Gamma_{0n}), \quad n \in \mathbb{N}, \tag{3.22}$$

solves Problem 3.4. *More precisely, $\widetilde{\Pi}_n$ is a boundary triplet for S_n^* such that $\widetilde{S}_{0n} := S_n^* \upharpoonright \ker(\widetilde{\Gamma}_{0n}) = S_{0n}$, $n \in \mathbb{N}$, and the direct sum $\widetilde{\Pi} = \{\mathcal{H}, \widetilde{\Gamma}_0, \widetilde{\Gamma}_1\} := \bigoplus_{n=1}^{\infty} \widetilde{\Pi}_n$ defined by* (3.3)*, forms a boundary triplet for A^*. Moreover, one has $\widetilde{M}(i) = iI_{\mathcal{H}}$ where $\widetilde{M}(z)$ is the Weyl function of $\widetilde{\Pi}$ given by* (3.14)*.*

Proof. By Lemma 3.5, for each $n \in \mathbb{N}$ the triplet $\widetilde{\Pi}_n$ is a boundary triplet for S_n^* such that $S_{0n} = S_n^* \upharpoonright \ker(\Gamma_{0n})$ and $\widetilde{M}_n(i) = iI_{\mathcal{H}_n}$. By Lemma 3.6, the mapping $\widetilde{\Gamma}_n := (\widetilde{\Gamma}_{0n}, \widetilde{\Gamma}_{1n}) : \mathfrak{H}_{+n} \to \mathcal{H}_n \oplus \mathcal{H}_n$, $n \in \mathbb{N}$, is contractive. Hence $\|\widetilde{\Gamma}_j\| = \sup_n \|\widetilde{\Gamma}_{jn}\| \leqslant 1$, where $\widetilde{\Gamma}_j$, $j \in \{0, 1\}$, is defined by (3.3). It follows that the mappings $\widetilde{\Gamma}_0$ and $\widetilde{\Gamma}_1$ are well defined on $\mathrm{dom}(A^*) = \bigoplus_{n=1}^{\infty} \mathrm{dom}(S_n^*)$. Thus, the Green's identity (3.5) holds for all $f, g \in \mathrm{dom}(A^*)$.

Further, we set $\mathfrak{N}_{\pm in} := \ker(S_n^* \mp i)$, $\mathfrak{N}_n := \mathfrak{N}_{in} \dot{+} \mathfrak{N}_{-in}$, $\mathfrak{N}_{\pm i} := \ker(A^* \mp i)$ and $\mathfrak{N} := \mathfrak{N}_i \dot{+} \mathfrak{N}_{-i}$. By Lemma 3.6, the restriction $\Gamma_n \upharpoonright \mathfrak{N}_n$ is an isometry from $\mathfrak{N}_n$, regarded as a subspace of $\mathfrak{H}_{+n}$, onto $\mathcal{H}_n \oplus \mathcal{H}_n$. Since $\mathfrak{N}$ regarded as a subspace of $\mathfrak{H}_+$ admits the representation $\mathfrak{N} = \bigoplus_{n=1}^{\infty} \mathfrak{N}_n$, the restriction $\Gamma \upharpoonright \mathfrak{N}$, $\Gamma := \bigoplus_{n=1}^{\infty} \Gamma_n$, isometrically maps $\mathfrak{N}$ onto $\mathcal{H} \oplus \mathcal{H}$. Hence $\mathrm{ran}(\Gamma) = \mathcal{H} \oplus \mathcal{H}$. Now equalities (3.14) are immediate from Definition 2.4. □

Let us apply the regularization procedure to Example 3.2.

Example 3.8. We use the notation of Example 3.2. Let S_n be the symmetric operator defined by (3.7) and let $\Pi_n = \{\mathbb{C}, \Gamma_{0n}, \Gamma_{1n}\}$ be the boundary triplet for S_n^* defined by (3.10). The corresponding Weyl function $M_n(z)$ is given by $M_n(z) = -\cot(2^{-1} z d_n)$.

If $d_*(X) = 0$, then the direct sum $\Pi := \bigoplus_{n=1}^{\infty} \Pi_n$ is not a boundary triplet for $A^* = \bigoplus_{n=1}^{\infty} S_n^*$, cf. Example 3.2. Let us regularize the boundary triplets Π_n, $n \in \mathbb{N}$ according to Theorem 3.7, i.e. applying formulas (3.22). One has

$$R_n^2 = \mathrm{Im}\big(M_n(i)\big) = \big(1 + e^{-2d_n}\big)\big(1 - e^{-2d_n}\big)^{-1} \quad \text{and} \quad Q_n = \mathrm{Re}\big(M_n(i)\big) = 0.$$

According to formulas (3.22) we set $\widetilde{\Pi}_n = \{\mathbb{C}, \widetilde{\Gamma}_{0n}, \widetilde{\Gamma}_{1n}\}$ where $\widetilde{\Gamma}_{0n} = R_n \Gamma_{0n}$ and $\widetilde{\Gamma}_{1n} = R_n^{-1} \Gamma_{1n}$, $n \in \mathbb{N}$. By Theorem 3.7, the direct sum $\widetilde{\Pi} := \bigoplus_{n=1}^{\infty} \widetilde{\Pi}_n$ is a boundary triplet for A^*. Clearly, $R_n^2 \sim d_n^{-1}$ whenever $d_n \to 0$ as $n \to \infty$. Therefore the triplets $\widetilde{\Pi}_n$ non-essentially differ from the triplets $\widehat{\Pi}_n$ defined in Example 3.2 directly, without regularization construction.

Note in conclusion that $\mathrm{Im}(M(i)) = \bigoplus_{n=1}^{\infty} \mathrm{Im}(M_n(i)) = -iM(i)$ is unbounded.

Remark 3.9. (i) Let $\{S_n\}_{n \in \mathbb{N}}$ be a sequence of pairwise unitarily equivalent closed symmetric operators. In this case another construction of boundary triplets Π_n for S_n^*, $n \in \mathbb{N}$, such that $\Pi = \bigoplus_{n \in \mathbb{N}} \Pi_n$ is a boundary triplet for $A^* = \bigoplus_{n \in \mathbb{N}} S_n^*$ has been proposed by A.N. Kochubej [26, Theorem 3].

(ii) Note that all results of the section including Theorem 3.7 can easily be extended to the case of non-densely defined symmetric operators.

(iii) Further development of Theorem 3.7 can be found in [27].

3.3. Direct sums of extremal extensions of non-negative symmetric operators and operators with a gap

Next we apply Theorem 3.7 to direct sums of non-negative symmetric operators as well as symmetric operators with a gap.

Recall, that for any non-negative symmetric operator A the set of its non-negative self-adjoint extensions $\mathrm{Ext}_A(0, \infty)$ is non-empty (see [1,25]). The set $\mathrm{Ext}_A(0, \infty)$ contains the Friedrichs (the biggest) extension A^F and the Krein (the smallest) extension A^K. These extensions are uniquely determined by the following extremal property in the class $\mathrm{Ext}_A(0,\infty)$:

$$\big(A^F+x\big)^{-1} \leqslant (\widetilde{A}+x)^{-1} \leqslant \big(A^K+x\big)^{-1}, \quad x>0,\ \widetilde{A}\in \mathrm{Ext}_A(0,\infty).$$

Corollary 3.10. *Assume the conditions of Theorem* 3.7. *Further, let $S_n \geqslant 0$, $n\in\mathbb{N}$, and let S_n^F and S_n^K be the Friedrichs and the Krein extensions of S_n, respectively. Then*

$$A^F=\bigoplus_{n=1}^{\infty} S_n^F \quad \textit{and} \quad A^K=\bigoplus_{n=1}^{\infty} S_n^K. \tag{3.23}$$

Proof. Let us prove the second relation. The first one is proved similarly. By Theorem 3.7 there exists a boundary triplet $\Pi_n=\{\mathcal{H}_n,\Gamma_{0n},\Gamma_{1n}\}$ for S_n^* such that $S_n^K=S_{0n}$ and $\Pi=\bigoplus_{n=1}^{\infty}\Pi_n$ is a boundary triplet for A^*.

Fix any $x_2\in\mathbb{R}_+$ and put $C_2:=\|M(-x_2)\|$. Then any $h=\bigoplus_{n=1}^{\infty}h_n\in\mathcal{H}$ can be decomposed by $h=h^{(1)}\oplus h^{(2)}$ with $h^{(1)}\in\bigoplus_{n=1}^{p}\mathcal{H}_n$ and $h^{(2)}\in\bigoplus_{n=p+1}^{\infty}\mathcal{H}_n$ such that $\|h^{(2)}\|<C_2^{-1/2}$. Hence $|(M(-x_2)h^{(2)},h^{(2)})|<1$. Due to the monotonicity of $M(\cdot)$ we get

$$\big(M(-x)h^{(2)},h^{(2)}\big)>\big(M(-x_2)h^{(2)},h^{(2)}\big)>-1, \quad x\in(0,x_2).$$

Since $S_{0n}=S_n^K$, the Weyl function $M_n(\cdot)$ satisfies

$$\lim_{x\downarrow 0}\big(M_n(-x)g_n,g_n\big)=+\infty, \quad g_n\in\mathcal{H}_n\setminus\{0\}, \tag{3.24}$$

cf. [13, Proposition 4]. Because $M(\cdot)=\bigoplus_{n=1}^{\infty}M_n(\cdot)$ is block-diagonal, cf. (3.14), we get from (3.24) that for any $N>0$ there exists $x_1>0$ such that

$$\big(M(-x)h^{(1)},h^{(1)}\big)=\sum_{n=1}^{p}\big(M_n(-x)h_n,h_n\big)>N \quad \text{for } x\in(0,x_1). \tag{3.25}$$

Combining (3.24) with (3.25) and using the diagonal form of $M(\cdot)$, we get

$$\big(M(-x)h,h\big)=\big(M(-x)h^{(1)},h^{(1)}\big)+\big(M(-x)h^{(2)},h^{(2)}\big)>N-1$$

for $0<x\leqslant\min(x_1,x_2)$. Thus, $\lim_{x\downarrow 0}(M(-x)h,h)=+\infty$ for $h\in\mathcal{H}\setminus\{0\}$. Applying [13, Proposition 4] we prove the second relation of (3.23). $\square$

Remark 3.11. Another proof can be obtained by using characterization of A^F and A^K by means of the respective quadratic forms.

Similar result is also valid for direct sums of symmetric operators with a finite gap. To state the corresponding result we recall that a symmetric operator S has a gap (α,β) if

$$\big\|\big(2S-(\alpha+\beta)\big)f\big\|\geqslant(\beta-\alpha)\|f\|, \quad f\in\mathrm{dom}(S).$$

According to the Krein result [28] the set $\mathrm{Ext}_S(\alpha,\beta)$ of all extensions $\widetilde{S}=\widetilde{S}^*\in\mathrm{Ext}_S$ preserving the gap is non-empty (see also [13]). Moreover, the set $\mathrm{Ext}_S(\alpha,\beta)$ contains two extremal extensions $S^{(\alpha)}$ and $S^{(\beta)}$

$$\big(S^{(\alpha)}-\lambda\big)^{-1}\leqslant(\widetilde{S}-\lambda)^{-1}\leqslant\big(S^{(\beta)}-\lambda\big)^{-1},\quad \lambda\in(\alpha,\beta),\ \widetilde{S}\in\mathrm{Ext}_{S(\alpha,\beta)}$$

(see [13, Section 4]). It follows that the class $\mathrm{Ext}_{S(\alpha,\beta)}$ contains only one element if and only if $S^{(\alpha)}=S^{(\beta)}$.

Corollary 3.12. *Assume the conditions of Theorem* 3.7. *Let also S_n, $n\in\mathbb{N}$, have a gap (α,β) and let $S_n^{(\alpha)}$, $S_n^{(\beta)}$ be the extremal extensions of S_n. Then*

$$A^{(\alpha)}=\bigoplus_{n=1}^{\infty}S_n^{(\alpha)}\quad\textit{and}\quad A^{(\beta)}=\bigoplus_{n=1}^{\infty}S_n^{(\beta)}.$$

The proof is similar to that of Corollary 3.10 and is based on Theorem 3.7 and the characterization of extremal extensions $S^{(\alpha)}$ and $S^{(\beta)}$ in terms of the Weyl function given in [13, Proposition 10].

3.4. Ac-spectrum of direct sums of symmetric operators with arbitrary deficiency indices

We start with some simple spectral observations for direct sums of symmetric operators where the symmetric operators may have arbitrary deficiency indices.

Proposition 3.13. *Let $\{S_n\}_{n=1}^{\infty}$ be a sequence of densely defined closed symmetric operators in $\mathfrak{H}_n$ and let $S_{0n}=S_{0n}^*\in\mathrm{Ext}_{S_n}$. Further, let A and A_0 be given by* (3.1) *and* (3.4)*, respectively. If $\widetilde{A}=\widetilde{A}^*$ is an extension of A satisfying*

$$(\widetilde{A}-i)^{-1}-(A_0-i)^{-1}\in\mathfrak{S}_\infty(\mathfrak{H})\tag{3.26}$$

then

$$\sigma_{ac}(A_0)=\overline{\bigcup\sigma_{ac}(S_{0n})}\subseteq\sigma(\widetilde{A})\quad\text{and}\quad\sigma_{ac}(\widetilde{A})\subseteq\overline{\bigcup\sigma(S_{0n})}=\sigma(A_0).\tag{3.27}$$

Proof. By the Weyl theorem, condition (3.26) yields $\sigma_{\mathrm{ess}}(\widetilde{A})=\sigma_{\mathrm{ess}}(A_0)$. Hence

$$\overline{\bigcup\sigma_{ac}(S_{0n})}=\sigma_{ac}(A_0)\subseteq\sigma_{\mathrm{ess}}(A_0)=\sigma_{\mathrm{ess}}(\widetilde{A})\subseteq\sigma(\widetilde{A})$$

and

$$\sigma_{ac}(\widetilde{A})\subseteq\sigma_{\mathrm{ess}}(\widetilde{A})=\sigma_{\mathrm{ess}}(A_0)\subseteq\sigma(A_0)=\overline{\bigcup\sigma(S_{0n})}$$

which completes the proof. □

Applying Theorem 2.11 we improve Proposition 3.13 as follows.

Theorem 3.14. *Assume the conditions of Proposition* 3.13. *Let also $\Pi_n=\{\mathcal{H}_n,\Gamma_{0n},\Gamma_{1n}\}$ be a boundary triplet for S_n^* such that $S_{0n}=S_n^*\upharpoonright\ker(\Gamma_{0n})$, $n\in\mathbb{N}$, and let $M_n(\cdot)$ be the corresponding Weyl function. Moreover, let $\mathfrak{m}_n^+(\cdot)$, $n\in\mathbb{N}$, be the invariant maximal normal function for Π_n and let δ be a Lebesgue measurable subset of $\mathbb{R}$ such that $\sup_{n\in\mathbb{N}}\mathfrak{m}_n^+(t)<+\infty$ for a.e. $t\in\delta$. Then the ac-parts $\widetilde{A}^{ac}E_{\widetilde{A}}(\delta)$ and $A_0^{ac}E_{A_0}(\delta)$ of $\widetilde{A}E_{\widetilde{A}}(\delta)$ and $A_0E_{A_0}(\delta)$, respectively, are unitarily equivalent. In particular, if $\delta=\mathbb{R}$, then the ac-parts $\widetilde{A}^{ac}$ and A_0^{ac} are unitarily equivalent and* (3.27) *is replaced by $\sigma_{ac}(A_0)=\sigma_{ac}(\widetilde{A})$.*

Proof. Let $\widetilde{\Pi}_n = \{\mathcal{H}_n, \widetilde{\Gamma}_{0n}, \widetilde{\Gamma}_{1n}\}$ be a boundary triplet for S_n^*, $n \in \mathbb{N}$, defined according to (3.22), that is $\widetilde{\Gamma}_{0n} := R_n \Gamma_{0n}$ and $\widetilde{\Gamma}_{1n} := R_n^{-1}(\Gamma_{1n} - \mathrm{Re}(M_n(i))\Gamma_{0n})$, where $R_n := \sqrt{\mathrm{Im}(M_n(i))}$. The corresponding Weyl function $\widetilde{M}_n(\cdot)$ is

$$\widetilde{M}_n(z) = R_n^{-1}\big(M_n(z) - \mathrm{Re}\, M_n(i)\big)R_n^{-1}, \quad n \in \mathbb{N}.$$

Since $\widetilde{M}_n(i) = iI_{\mathcal{H}_n}$, $n \in \mathbb{N}$, by Theorem 3.7, $\widetilde{\Pi} = \bigoplus_{n=1}^{\infty} \widetilde{\Pi}_n =: \{\mathcal{H}, \widetilde{\Gamma}_0, \widetilde{\Gamma}_1\}$ is a boundary triplet for $A^* = \bigoplus_{n=1}^{\infty} S_n^*$ satisfying $A^* \upharpoonright \ker \widetilde{\Gamma}_0 = A_0 := \bigoplus_{n=1}^{\infty} S_{0n}$. By the definition of $\mathfrak{m}_n^+(\cdot)$, one has $\mathfrak{m}_n^+(t) = \widetilde{m}_n^+(t) := \sup_{y \in (0,1]} \|\widetilde{M}_n(t+iy)\|$ for $t \in \mathbb{R}$, $n \in \mathbb{N}$. Since $A_0 = \bigoplus_{n=1}^{\infty} S_{0n}$ we get that $\widetilde{m}^+(t) = \sup_n m_n^+(t)$, where $\widetilde{m}^+(t) := \sup_{y\in(0,1]} \|\widetilde{M}(t+iy)\|$, $t \in \mathbb{R}$. By assumption, the maximal normal function $\widetilde{m}^+(t)$ is finite for a.e. $t \in \delta$. Hence Theorem 2.11 yields the unitary equivalence of $\widetilde{A}^{ac} E_{\widetilde{A}}(\delta)$ and $A_0^{ac} E_{A_0}(\delta)$. $\square$

Let T and T' be densely defined closed symmetric operators in $\mathfrak{H}$ and let T_0 and T_0' be self-adjoint extensions of T and T', respectively. The pairs $\{T, T_0\}$ and $\{T', T_0'\}$ are called unitarily equivalent if there exists a unitary operator U in $\mathfrak{H}$ such that $T' = UTU^{-1}$ and $T_0' = UT_0U^{-1}$.

Corollary 3.15. *Let the assumptions of Theorem* 3.14 *be satisfied. Moreover, let the pairs* $\{S_n, S_{0n}\}$, $n \in \mathbb{N}$, *be unitarily equivalent to the pair* $\{S_1, S_{01}\}$. *Assume that the maximal normal function* $m_1^+(t) := \sup_{0<y\leqslant 1} \|M_1(t+iy)\|$ *is finite for a.e.* $t \in \delta$ *and the condition* (3.26) *is satisfied. Then the ac-parts* $\widetilde{A}^{ac} E_{\widetilde{A}}(\delta)$ *and* $A_0^{ac} E_{A_0}(\delta)$ *are unitarily equivalent.*

Proof. Since the symmetric operators S_n are unitarily equivalent, we assume without loss of generality that $\mathcal{H}_n = \mathcal{H}$ for each $n \in \mathbb{N}$. Let U_n be a unitary operator such that $A_1 = U_n S_n U_n^{-1}$ and $A_{01} = U_n S_{0n} U_n^{-1}$. A straightforward computation shows that $\Pi_n' := \{\mathcal{H}, \Gamma_{0n}', \Gamma_{1n}'\}$, $\Gamma_{0n}' := \Gamma_{01} U_n$ and $\Gamma_{1n}' := \Gamma_{1n} U_n$, defines a boundary triplet for S_n^*. The Weyl function $M_n'(\cdot)$ corresponding to Π_n' is $M_n'(z) = M_1(z)$. Hence $\mathfrak{m}_n^+(\cdot) = \mathfrak{m}_n'^+(\cdot)$ and $\mathfrak{m}_1^+(t) = \mathfrak{m}_n'^+(t)$ for $t \in \mathbb{R}$, where $\mathfrak{m}_n^+(t)$ and $\mathfrak{m}_n'^+(t)$ are the invariant maximal normal functions corresponding to the triplets Π_n and Π_n', respectively. Since $\mathfrak{m}_1^+(t) = \mathfrak{m}_n^+(t)$ for $t \in \mathbb{R}$ and $n \in \mathbb{N}$ one completes the proof by applying Theorem 3.14. $\square$

3.5. Ac-spectrum of direct sums of symmetric operators with finite deficiency indices

Here we improve the previous results assuming that $n_\pm(S_n) < \infty$. First, we show that extensions of the form (3.4) possess the property of *ac*-minimality. To this end we start with the following lemma.

Lemma 3.16. *Let* H *be a bounded non-negative self-adjoint operator in a separable Hilbert space* $\mathfrak{H}$ *and let* L *be a bounded operator in* $\mathfrak{H}$. *Then one has*:

(i) $\dim(\overline{\mathrm{ran}(H)}) = \dim(\overline{\mathrm{ran}(\sqrt{H})})$.
(ii) *If* $L^*L \leqslant H$, *then* $\dim(\overline{\mathrm{ran}(L)}) \leqslant \dim(\overline{\mathrm{ran}(H)})$.
(iii) *If* P *is an orthogonal projection, then* $\dim(\overline{\mathrm{ran}(PHP)}) \leqslant \dim(\overline{\mathrm{ran}(H)})$.

Proof. The assertion (i) is obvious.

(ii) If $L^*L \leqslant H$, then there is a contraction C such that $L = C\sqrt{H}$. Hence $\dim(\overline{\mathrm{ran}(L)}) = \dim(\overline{\mathrm{ran}(C\sqrt{H})}) \leqslant \dim(\overline{\mathrm{ran}(\sqrt{H})}) = \dim(\overline{\mathrm{ran}(H)})$.

(iii) Clearly, $\dim(\overline{\mathrm{ran}(PHP)}) \leqslant \dim(\overline{\mathrm{ran}(\sqrt{H}P)}) \leqslant \dim(\overline{\mathrm{ran}(\sqrt{H})})$. Applying (i) we complete the proof. $\square$

Theorem 3.17. *Let* $\{S_n\}_{n=1}^{\infty}$ *be a sequence of densely defined closed symmetric operators in* $\mathfrak{H}_n$ *and let* $S_{0n} = S_{0n}^* \in \mathrm{Ext}_{S_n}$. *Let also* A *and* A_0 *be given by* (3.1) *and* (3.4), *respectively. If* $n_\pm(S_n) < \infty$ *for each* $n \in \mathbb{N}$, *then* A_0

is ac-minimal, i.e. $N_{A_0^{ac}}(t) \leqslant N_{\widetilde{A}^{ac}}(t)$ *for a.e.* $t \in \mathbb{R}$ *and any* $\widetilde{A} = \widetilde{A}^* \in \mathrm{Ext}_A$. *In particular,* $\sigma_{ac}(A_0) \subseteq \sigma_{ac}(\widetilde{A})$ *for any* $\widetilde{A} = \widetilde{A}^* \in \mathrm{Ext}_A$.

Proof. By Theorem 3.7 there exists a sequence of boundary triplets $\Pi_n := \{\mathcal{H}_n, \Gamma_{0n}, \Gamma_{1n}\}$, $n \in \mathbb{N}$, for S_n^* such that $S_{0n} = S_n^* \upharpoonright \ker(\Gamma_{0n})$, $n \in \mathbb{N}$, and the direct sum $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\} = \bigoplus_{n=1}^{\infty} \Pi_n$ is a boundary triplet for A^* satisfying $A_0 = A^* \upharpoonright \ker(\Gamma_0)$. By Proposition 2.3, any $\widetilde{A} = \widetilde{A}^* \in \mathrm{Ext}_A$ admits a representation $\widetilde{A} = A_\Theta$ with $\Theta = \Theta^* \in \widetilde{\mathcal{C}}(\mathcal{H})$. By [38, Corollary 4.2(i)], we can assume that $\widetilde{A}$ and A_0, are disjoint, that is $\Theta = B = B^* \in \mathcal{C}(\mathcal{H})$. Consider the generalized Weyl function $M_B(\cdot) := (B - M(\cdot))^{-1}$, where $M(\cdot) = \bigoplus_{n=1}^{\infty} M_n(\cdot)$, cf. (3.14). Clearly,

$$\mathrm{Im}\big(M_B(z)\big) = M_B(z)^* \,\mathrm{Im}\big(M(z)\big) M_B(z), \quad z \in \mathbb{C}_+.$$

Denote by P_N, $N \in \mathbb{N}$, the orthogonal projection from $\mathcal{H}$ onto the subspace $\mathcal{H}_N := \bigoplus_{n=1}^{N} \mathcal{H}_n$. Setting $M_B^{P_N}(z) := P_N M_B(z) \upharpoonright \mathcal{H}_N$, and taking into account the block-diagonal form of $M(\cdot)$ and the inequality $\mathrm{Im}(M(z)) > 0$ we obtain

$$\begin{aligned} \mathrm{Im}\big(M_B^{P_N}(z)\big) &= \mathrm{Im}\big(P_N M_B(z) P_N\big) \\ &= P_N M_B(z)^* \,\mathrm{Im}\big(M(z)\big) M_B(z) P_N \geqslant M_B^{P_N}(z)^* \,\mathrm{Im}\big(M^{P_N}(z)\big) M_B^{P_N}(z), \end{aligned} \tag{3.28}$$

where $M^{P_N}(z) := P_N M(z) \upharpoonright \mathcal{H}_N = \bigoplus_{n=1}^{N} M_n(z)$. Since P^N is a finite dimensional projection the limits $M_B^{P_N}(t) := \text{s-lim}_{y\downarrow 0} M_B^{P_N}(t+iy)$ and $M^{P_N}(t) := \text{s-lim}_{y\downarrow 0} M^{P_N}(t+iy)$ exist for a.e. $t \in \mathbb{R}$. From (3.28) we get

$$\mathrm{Im}\big(M_B^{P_N}(t)\big) \geqslant M_B^{P_N}(t)^* \,\mathrm{Im}\big(M^{P_N}(t)\big) M_B^{P_N}(t) \quad \text{for a.e. } t \in \mathbb{R}. \tag{3.29}$$

Since $M_B(\cdot)$ is a generalized Weyl function, it is a strict $R_{\mathcal{H}}$-function, that is, $\ker(\mathrm{Im}(M_B(z))) = \{0\}$, $z \in \mathbb{C}_+$. Therefore, $M_B^{P_N}(\cdot)$ is also strict. Hence $0 \in \varrho(M_B^{P_N}(z))$, $z \in \mathbb{C}_+$, and $G_N(\cdot) := -(M_B^{P_N}(\cdot))^{-1}$ is strict. Since both $G_N(\cdot)$ and $M_B^{P_N}(\cdot)$ are matrix-valued R-functions, the limits $M_B^{P_N}(t+i0) := \lim_{y\downarrow 0} M_B^{P_N}(t+iy)$ and $G_N(t+i0) := \lim_{y\downarrow 0} G_N(t+iy)$ exist for a.e. $t \in \mathbb{R}$. Therefore, passing to the limit in the identity $M_B^{P_N}(t+iy)G_N(t+iy) = -I$ as $y \to 0$, we get $M_B^{P_N}(t+i0)G_N(t+i0) = -I$ for a.e. $t \in \mathbb{R}$. Hence $M_B^{P_N}(t) := M_B^{P_N}(t+i0)$ is invertible for a.e. $t \in \mathbb{R}$.

Further, combining (3.29) with Lemma 3.16(ii) we get

$$\dim\big(\overline{\mathrm{ran}\big(\sqrt{\mathrm{Im}\, M^{P_N}(t)}\, M_B^{P_N}(t)\big)}\big) \leqslant d_{M_B^{P_N}}(t) \quad \text{for a.e. } t \in \mathbb{R}.$$

Since $M_B^{P_N}(t)$ is invertible for a.e. $t \in \mathbb{R}$, we find

$$d_{M^{P_N}}(t) := \dim\big(\overline{\mathrm{ran}\big(\sqrt{\mathrm{Im}\, M^{P_N}(t)}\big)}\big) \leqslant d_{M_B^{P_N}}(t) \quad \text{for a.e. } t \in \mathbb{R}. \tag{3.30}$$

Let $D_N = P_N \oplus D_0$ where $D_0 \in \mathfrak{S}_2(\mathcal{H}_N^\perp)$ and satisfy $\ker(D_0) = \ker(D_0^*) = \{0\}$. Then $\ker(D_N) = \ker(D_N^*) = \{0\}$ and $P_N = P_N D_N = D_N P_N$. By Lemma 3.16(iii), $d_{M^{P_N}}(t) \leqslant d_{M_B^{D_N}}(t)$ for a.e. $t \in \mathbb{R}$. Further, for any $D \in \mathfrak{S}_2(\mathcal{H})$ and satisfying $\ker(D) = \ker(D^*) = \{0\}$, $d_{M_B^D}(t) = d_{M_B^{D_N}}(t)$ for a.e. $t \in \mathbb{R}$. Combining this equality with (3.30) we get $d_{M^{P_N}}(t) \leqslant d_{M_B^D}(t)$ for a.e. $t \in \mathbb{R}$ and $N \in \mathbb{N}$. Since

$$d_{M^{P_N}}(t) = \sum_{n=1}^{N} d_{M_n(t)} \quad \text{and} \quad d_{M^D}(t) = \sum_{n=1}^{\infty} d_{M_n}(t) \tag{3.31}$$

for a.e. $t \in \mathbb{R}$, we finally prove that $d_{M^D}(t) \leqslant d_{M_B^D}(t)$ for a.e. $t \in \mathbb{R}$. One completes the proof by applying Proposition 2.10(i). $\square$

Corollary 3.18. *Let the assumptions of Theorem* 3.17 *be satisfied. If $S_n \geqslant 0$, $n \in \mathbb{N}$ and if the deficiency indices of S_n are finite for each $n \in \mathbb{N}$, then the Friedrichs and the Krein extensions A^F and A^K of A are ac-minimal. In particular, $(A^F)^{ac}$ and $(A^K)^{ac}$ are unitarily equivalent.*

Proof. Combining Theorem 3.17 with Corollary 3.10 one gets the result. $\square$

Corollary 3.19. *Assume the conditions of Theorem* 3.14*. Further, let $n_\pm(S_n) < \infty$ for each $n \in \mathbb{N}$ and let $\widetilde{A} = \widetilde{A}^* \in \mathrm{Ext}_A$.*

(i) *If*

$$\delta_\infty := \left\{ t \in \mathbb{R}\colon \sum_{n\in\mathbb{N}} d_{M_n}(t) = \infty \right\}, \tag{3.32}$$

then the ac-parts $\widetilde{A}^{ac} E_{\widetilde{A}}(\delta_\infty)$ and $A_0^{ac} E_{A_0}(\delta_\infty)$ are unitarily equivalent.

(ii) *If δ is a Lebesgue measurable subset of $\mathbb{R}$ such that $\sup_n \mathfrak{m}_n^+(t) < \infty$ for a.e. $t \in \delta$, then the ac-parts $\widetilde{A}^{ac} E_{\widetilde{A}}(\delta_\infty \cup \delta)$ and $A_0^{ac} E_{A_0}(\delta_\infty \cup \delta)$ are unitarily equivalent.*

Proof. (i) By (3.31) and (3.32) we find $d_{M^D}(t) = +\infty$ for a.e. $t \in \delta_\infty$. Since by Theorem 3.17, A_0 is *ac*-minimal, one gets that $N_{\widetilde{A}^{ac}}(t) = N_{A_0^{ac}}(t)$ for a.e. $t \in \delta_\infty$. This yields the unitary equivalence of the *ac*-parts $\widetilde{A}^{ac} E_{\widetilde{A}}(\delta_\infty)$ and $A_0^{ac} E_{A_0}(\delta_\infty)$.

(ii) By Theorem 3.14 the *ac*-parts $\widetilde{A}^{ac} E_{\widetilde{A}}(\delta)$ and $A_0^{ac} E_{A_0}(\delta)$ are unitarily equivalent. Using (i) we immediately obtain the unitary equivalence of the parts $\widetilde{A}^{ac} E_{\widetilde{A}}(\delta_\infty \cup \delta)$ and $A_0^{ac} E_{A_0}(\delta_\infty \cup \delta)$. $\square$

Corollary 3.20. *Let the assumptions of Theorem* 3.17 *be satisfied. If the deficiency indices of S_n are finite for each $n \in \mathbb{N}$, then $\overline{\bigcup_{n\in\mathbb{N}} \sigma_{ac}(S_{0n})} \subseteq \sigma_{ac}(\widetilde{A})$ for any self-adjoint extension $\widetilde{A}$ of A. If in addition condition* (3.26) *is valid and the extensions S_{0n} are purely absolutely continuous for each $n \in \mathbb{N}$, then*

$$\sigma_{ac}(\widetilde{A}) = \overline{\bigcup_{n\in\mathbb{N}} \sigma_{ac}(S_{0n})}. \tag{3.33}$$

Proof. The first statement immediately follows from Theorem 3.17. Relation (3.33) is implied by Proposition 3.13. $\square$

Corollary 3.21. *Let the assumptions of Theorem* 3.17 *be satisfied. Further, let the pairs $\{S_n, S_{0n}\}$, $n \in \mathbb{N}$, be unitarily equivalent to $\{S_1, S_{01}\}$. If the deficiency indices of S_n are finite for each $n \in \mathbb{N}$, then for any $\widetilde{A} = \widetilde{A}^* \in \mathrm{Ext}_A$ satisfying condition* (3.26) *the ac-parts $\widetilde{A}^{ac}$ and A_0^{ac} are unitarily equivalent.*

Proof. The proof follows immediately from Corollary 3.15. $\square$

Remark 3.22. (i) For the special case $n_\pm(S_n) = 1$, $n \in \mathbb{N}$, Theorem 3.17 complements [2, Corollary 5.4] where the inclusion $\sigma_{ac}(A_0) \subseteq \sigma_{ac}(\widetilde{A})$ was proved. Moreover, Corollary 3.21 might be regarded as a substantial generalization of [2, Theorem 5.6(i)] to the case $n_\pm(S_n) > 1$. However, in the case $n_\pm(S_n) = 1$, Corollary 3.21 is implied by [2, Theorem 5.6(i)] where the unitary equivalence of $\widetilde{A}^{ac} = \widetilde{A}_B^{ac}$ and A_0^{ac} was proved under the weaker assumption that B is purely singular. Indeed, by Proposition 2.7 condition (3.26) with $\widetilde{A} = A_B$ is equivalent to the discreteness of B.

(ii) The inequality $N_{A_0^{ac}}(t) \leqslant N_{\widetilde{A}^{ac}}(t)$ in Theorem 3.17 might be strict even for $t \in \sigma_{ac}(A_0)$. Indeed, assume that (α, β) is a gap for all except for the operators $S_1, \ldots, S_N$. Set $S_N := \bigoplus_{n=1}^N S_n$ and $S_\infty := \bigoplus_{n=N+1}^\infty S_n$. Then $n_\pm(S_\infty) = \infty$ and (α, β) is a gap for S_∞. By [9] there exists an extension $\widetilde{S}_\infty = \widetilde{S}_\infty^* \in \mathrm{Ext}_{A_2}$ having *ac*-spectrum within (α, β) of arbitrary multiplicity.

Moreover, even for operators S_n satisfying the assumptions of Corollary 3.21 with $n_\pm(S_n) = 1$ the inclusion $\sigma_{ac}(A_0) \subseteq \sigma_{ac}(\widetilde{A})$ might be strict if condition (3.26) is violated, cf. [9] or [2, Theorem 4.4] which guarantees the appearance of prescribed spectrum either within one gap or within several gaps of A_0.

4. Sturm–Liouville operators with bounded operator potentials

4.1. The Dirichlet, the Neumann and the Krein realizations

Let $\mathcal{H}$ be an infinite dimensional separable Hilbert space. As usual, $L^2(\mathbb{R}_+, \mathcal{H})$ stands for the Hilbert space of (weakly) measurable vector-functions $f(\cdot): \mathbb{R}_+ \to \mathcal{H}$ satisfying $\int_{\mathbb{R}_+} \|f(t)\|^2_{\mathcal{H}}\, dt < \infty$. Denote also by $W^{2,2}(\mathbb{R}_+, \mathcal{H})$ the Sobolev space of vector-functions taking values in $\mathcal{H}$.

Let $T = T^* \geqslant 0$ be a bounded operator in $\mathcal{H}$. Denote by $A := A_{\min}$ the minimal operator generated by $\mathcal{A}$, cf. (1.1), in $\mathfrak{H} := L^2(\mathbb{R}_+, \mathcal{H})$. It is known (see [20,40]) that the minimal operator A is given by

$$(Af)(x) = -\frac{d^2}{dx^2} f(x) + Tf(x), \quad f \in \mathrm{dom}(A) = W_0^{2,2}(\mathbb{R}_+, \mathcal{H}), \tag{4.1}$$

where $W_0^{2,2}(\mathbb{R}_+, \mathcal{H}) := \{f \in W^{2,2}(\mathbb{R}_+, \mathcal{H}):\ f(0) = f'(0) = 0\}$.

The operator A is closed, symmetric and non-negative. The adjoint operator A^* is given by [20, Theorem 3.4.1]

$$\big(A^* f\big)(x) = -\frac{d^2}{dx^2} f(x) + Tf(x), \quad f \in \mathrm{dom}\big(A^*\big) = W^{2,2}(\mathbb{R}_+, \mathcal{H}). \tag{4.2}$$

The deficiency subspace $\mathfrak{N}_z(A) = \ker(A^* - z)$ is given by

$$\mathfrak{N}_z(A) = \big\{g_z(x) := e^{ix\sqrt{z-T}} h:\ h \in \mathcal{H}\big\}, \quad z \in \mathbb{C}_\pm, \tag{4.3}$$

where a branch of the multifunction $\sqrt{\cdot}$ in $\mathbb{C}$ with the cut along $\mathbb{R}_+$ is fixed by $\sqrt{1 + i0} = 1$.

Lemma 4.1. *Let $T = T^* \geqslant 0$ be a bounded. Then the symmetric operator A given by* (4.1) *is simple.*

Proof. Without loss of generality we can assume that T has simple spectrum. Otherwise T is a direct sum of operators $T_n = T_n^* \geqslant 0$ with simple spectrum, $T = \bigoplus_{n=1}^\infty T_n$. Hence the operator A admits the direct sum decomposition $A = \bigoplus_{n=1}^\infty A_n$ where A_n is defined by (4.1) with T_n in place of T. To prove simplicity of A it suffices to prove the implication $f \in \mathfrak{H}$, $(f, g_z) = 0$, $g_z \in \mathfrak{N}_z(A)$, $z \in \mathbb{C} \setminus \mathbb{R} \Longrightarrow f = 0$. Since T has simple spectrum it is unitarily equivalent to the operator Q of multiplication by the independent variable λ in $L^2(\Delta, d\mu(\lambda))$, where $\Delta = [0, \|T\|]$ and μ is a finite Borel measure on Δ. Thus, we assume that $T = Q$, $\mathcal{H} := L^2(\Delta, d\mu(\lambda))$ and $\mathfrak{H} = L^2(\mathbb{R}_+, \mathcal{H}) = L^2(\mathbb{R}_+ \times \Delta, dx \otimes d\mu(\lambda))$. Let

$$0 = \int_{\mathbb{R}_+} dx \int_\Delta d\lambda\, f(x, \lambda) \overline{e^{i\sqrt{z-\lambda}x} h(\lambda)}, \quad z \in \mathbb{C} \setminus \mathbb{R},$$

$h(\cdot) \in L^2(\Delta, d\mu(\lambda))$, $f(\cdot,\cdot) \in L^2(\mathbb{R}_+ \times \Delta, dx \otimes d\mu(\lambda))$. By Fubini's theorem,

$$0 = \int_\Delta d\lambda\, \overline{h(\lambda)} \int_{\mathbb{R}_+} dx\, e^{-i\overline{\sqrt{z-\lambda}}x} f(x,\lambda), \quad z \in \mathbb{C} \setminus \mathbb{R}.$$

Since $h(\cdot)$ is arbitrary we find

$$0 = \int_{\mathbb{R}_+} dx\, e^{-sx} f(x,\lambda), \quad s := i\overline{\sqrt{z-\lambda}},$$

for μ-a.e. $\lambda \in \Delta$. If z runs through $\mathbb{C} \setminus \mathbb{R}$, then s runs through right complex plane $\mathbb{C}_{\text{right}}$. By the injectivity of the Laplace transform, $f(x,\lambda) = 0$ for a.e. $(x,\lambda) \in \mathbb{R}_+ \times \Delta$ with respect to $dx \otimes d\mu(\lambda)$. $\square$

The Dirichlet realization A^D is defined by $A^D f := \mathcal{A}f$, $f \in \operatorname{dom}(A^D) := \{g \in W^{2,2}(\mathbb{R}_+, \mathcal{H})\colon g(0) = 0\}$. Similarly, the Neumann realization A^N is defined by $A^N f := \mathcal{A}f$, $f \in \operatorname{dom}(A^N) := \{g \in W^{2,2}(\mathbb{R}_+, \mathcal{H})\colon g'(0) = 0\}$. Since $\operatorname{dom}(A) \subseteq \operatorname{dom}(A^D)$, $\operatorname{dom}(A^N) \subseteq \operatorname{dom}(A^*)$, one gets that A^D and A^N are proper extensions of A. Clearly, A^D and A^N are symmetric.

By [34, Theorem 1.3.1] the trace operators $\Gamma_0, \Gamma_1 : \operatorname{dom}(A^*) \to \mathcal{H}$,

$$\Gamma_0 f = f(0) \quad \text{and} \quad \Gamma_1 f = f'(0), \quad f \in \operatorname{dom}\bigl(A^*\bigr), \tag{4.4}$$

are well defined.

Lemma 4.2. *A triplet $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$, where Γ_0 and Γ_1 are defined by* (4.4)*, forms a boundary triplet for A^*. The corresponding Weyl function $M(\cdot)$ is*

$$M(z) = i\sqrt{z-T} = i\int_{\mathbb{R}_+} \sqrt{z-\lambda}\, dE_T(\lambda), \quad z \in \mathbb{C}_+. \tag{4.5}$$

Proof. One obtains the Green formula integrating by parts. The surjectivity of the mapping $\Gamma := (\Gamma_0, \Gamma_1) : \operatorname{dom}(A^*) \to \mathcal{H} \oplus \mathcal{H}$ follows from (4.4) and [34, Theorem 1.3.2]. Formula (4.5) is implied by (4.3). $\square$

Lemma 4.3. *Let T be a bounded non-negative self-adjoint operator in $\mathcal{H}$ and let A and $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be defined by* (4.1) *and* (4.4)*, respectively. Then*:

(i) *The invariant maximal normal function $\mathfrak{m}^+(t)$ of the Weyl function $M(\cdot)$ is finite for all $t \in \mathbb{R}$ and satisfies*

$$\mathfrak{m}^+(t) \leqslant (1+\sqrt{2})\bigl(1+t^2\bigr)^{1/4}, \quad t \in \mathbb{R}. \tag{4.6}$$

(ii) *The limit $M(t+i0) := \text{s-lim}_{y\downarrow 0} M(t+iy)$ exists, is bounded and equals*

$$M(t+i0) = i\int_{\mathbb{R}_+} \sqrt{t-\lambda}\, dE_T(\lambda) \quad \textit{for any } t \in \mathbb{R}. \tag{4.7}$$

(iii) $d_M(t) = \dim(\operatorname{ran}(E_T([0,t))))$ *for any $t \in \mathbb{R}$.*

Proof. (i) It follows from (4.5) and definition (2.11) that

$$\mathfrak{m}^+(t) \leqslant \sup_{y\in(0,1]}\sup_{\lambda\geqslant 0}\left|\frac{\sqrt{t+iy-\lambda}-\mathrm{Re}(\sqrt{i-\lambda}\,)}{\mathrm{Im}(\sqrt{i-\lambda}\,)}\right|.$$

Clearly, $\sqrt{i-\lambda} = (1+\lambda^2)^{1/4}e^{i(\pi-\varphi)/2}$ where $\varphi := \arccos(\frac{\lambda}{\sqrt{1+\lambda^2}})$. Hence

$$\left|\frac{\mathrm{Re}(\sqrt{i-\lambda}\,)}{\mathrm{Im}(\sqrt{i-\lambda}\,)}\right| = \tan\left(\frac{\varphi}{2}\right) = \frac{1}{\lambda+\sqrt{1+\lambda^2}} \leqslant 1, \quad \lambda \geqslant 0.$$

Furthermore, we have

$$\left|\frac{\sqrt{t+iy-\lambda}}{\mathrm{Im}(\sqrt{i-\lambda}\,)}\right| \leqslant \sqrt{2}\sqrt{\frac{\sqrt{(\lambda-t)^2+y^2}}{\lambda+\sqrt{1+\lambda^2}}} \leqslant \sqrt{2}\big(1+t^2\big)^{1/4}$$

for $\lambda \geqslant 0$, $t \in \mathbb{R}$ and $y \in (0,1]$ which yields (4.6).

(ii) From (4.5) we find $M(t) := M(t+i0) := \text{s-lim}_{y\downarrow 0}\, i\sqrt{t+iy-T} = i\sqrt{t-T}$, for any $t \in \mathbb{R}$, which proves (4.7). Clearly, $M(t) \in [\mathcal{H}]$ since $T \in [\mathcal{H}]$.

(iii) It follows that $\mathrm{Im}(M(t)) = \sqrt{t-T}\,E_T([0,t))$, which yields $d_M(t) = \dim(\mathrm{ran}(\mathrm{Im}(M(t)))) = \dim(\mathrm{ran}(E_T([0,t))))$. □

With $A = A_{\min}$ one associates a closable quadratic form $\mathfrak{t}'_F[f] := (Af, f)$, $\mathrm{dom}(\mathfrak{t}') = \mathrm{dom}(A)$. Its closure $\mathfrak{t}_F$ is given by

$$\mathfrak{t}_F[f] := \int_{\mathbb{R}_+} \big\{\big\|f'(x)\big\|^2_{\mathcal{H}} + \big\|\sqrt{T}f(x)\big\|^2_{\mathcal{H}}\big\}\,dx, \tag{4.8}$$

$f \in \mathrm{dom}(\mathfrak{t}_F) = W^{1,2}_0(\mathbb{R}_+,\mathcal{H})$, where $W^{1,2}_0(\mathbb{R}_+,\mathcal{H}) := \{f \in W^{1,2}(\mathbb{R}_+,\mathcal{H})\colon\ f(0) = 0\}$. By definition, the Friedrichs extension A^F of A is a self-adjoint operator associated with $\mathfrak{t}_F$. Clearly, $A^F = A^* \upharpoonright (\mathrm{dom}(A^*) \cap \mathrm{dom}(\mathfrak{t}_F))$.

Theorem 4.4. *Let $T \geqslant 0$, $T = T^* \in [\mathcal{H}]$, and $t_0 := \inf\sigma(T)$. Let A be defined by* (4.1) *and let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be the boundary triplet for A^* defined by* (4.4)*. Then the following hold*:

(i) *The Dirichlet realization A^D coincides with $A_0 := A^* \upharpoonright \ker(\Gamma_0)$ as well as with the Friedrichs extension A^F of A, $A^F = A^D$. Moreover, A^D is absolutely continuous and its spectrum is given by $\sigma(A^D) = \sigma_{ac}(A^D) = [t_0,\infty)$.*

(ii) *The Neumann realization A^N coincides with $A_1 := A^* \upharpoonright \ker(\Gamma_1)$. A^N is absolutely continuous $(A^N)^{ac} = A^N$ and $\sigma(A^N) = \sigma_{ac}(A^N) = [t_0,\infty)$.*

(iii) *The Krein realization* (*the Krein extension*) *A^K is given by*

$$\mathrm{dom}\big(A^K\big) = \big\{f \in W^{2,2}(\mathbb{R}_+,\mathcal{H})\colon\ f'(0) + \sqrt{T}f(0) = 0\big\}. \tag{4.9}$$

Moreover, $\ker(A^K) = \mathfrak{H}_0 := \overline{\{e^{-x\sqrt{T}}h\colon\ h \in \mathrm{ran}(T^{1/4})\}}$ and the restriction $A^K \upharpoonright \mathrm{dom}(A^K) \cap \mathfrak{H}_0^{\perp}$ is absolutely continuous, i.e. $\mathfrak{H}_0^{\perp} = \mathfrak{H}^{ac}(A^K)$ and $A^K = 0_{\mathfrak{H}_0} \oplus (A^K)^{ac}$. In particular, $\sigma(A^K) = \{0\} \cup \sigma_{ac}(A^K)$ and $\sigma_{ac}(A^K) = [t_0,\infty)$.

(iv) *The realizations A^D, A^N and $(A^K)^{ac}$ are unitarily equivalent.*

Proof. (i) It follows from (4.2) and (4.4) that $\mathrm{dom}(A^D) = \mathrm{dom}(A_0)$ which yields $A^D = A_0$. Since $\mathrm{dom}(A_0) \subseteq W_0^{1,2}(\mathbb{R}_+, \mathcal{H}) = \mathrm{dom}(\mathfrak{t}_F)$ we have $A^F = A_0$ (see [1, Section 8] and [25, Theorem 6.2.11]). It follows from (4.7) and [10, Theorem 4.3] that $\sigma_p(A_0) = \sigma_{sc}(A_0) = \emptyset$. Hence A_0 is absolutely continuous. Taking into account Lemma 4.3(iii) and Proposition 2.8 we get $\sigma(A_0) = \sigma_{ac}(A_0) = \mathrm{cl}_{ac}(\mathrm{supp}(d_M)) = [t_0, \infty)$ which proves (i).

(ii) Obviously we have $\mathrm{dom}(A^N) = \mathrm{dom}(A_1) := \ker(\Gamma_1)$ which proves $A^N = A_1$. It follows from Lemma 4.2 and (2.8) that the Weyl function corresponding to A_1 is given by

$$M_0(z) := \big(0 - M(z)\big)^{-1} = i(z-T)^{-1/2} = i \int \frac{1}{\sqrt{z-\lambda}}\, dE_T(\lambda), \quad z \in \mathbb{C}_+. \tag{4.10}$$

Since $M_0(\cdot)$ is regular within $(-\infty, t_0)$, we have $(-\infty, t_0) \subset \varrho(A_1)$. Further, let $\tau > t_0$. We set $\mathcal{H}_\tau := E_T([t_0, \tau))\mathcal{H}$ and note that for any $h \in \mathcal{H}_\tau$ and $t > \tau$

$$\big(M_0(t+i0)h, h\big) = i\big((t-T)^{-1/2}h, h\big) = i \int_{t_0}^{\tau} \frac{1}{\sqrt{t-\lambda}}\, d\big(E_T(\lambda)h, h\big). \tag{4.11}$$

Hence for any $h \in \mathcal{H}_\tau \setminus \{0\}$ and $t > \tau$ we have

$$0 < (t-t_0)^{-1/2}\|h\|^2 \leqslant \mathrm{Im}\big(M_0(t+i0)h, h\big) = \int_{t_0}^{\tau} (t-\lambda)^{-1/2}\, d\big(E_T(\lambda)h, h\big) < \infty.$$

By [10, Proposition 4.2], $\sigma_{ac}(A_1) \supseteq [\tau, \infty)$ for any $\tau > t_0$, which yields $\sigma_{ac}(A_1) = [t_0, \infty)$. It remains to show that A_1 is purely absolutely continuous. Since $M_0(t+i0) \notin [\mathcal{H}]$ we cannot apply [10, Theorem 4.3]. Fortunately, we can use [10, Corollary 4.7]. For any $t \in \mathbb{R}$, $y > 0$, and $h \in \mathcal{H}$ we set

$$V_h(t+iy) := \mathrm{Im}\big(M_0(t+iy)h, h\big) = \int \mathrm{Im}\left(\frac{1}{\sqrt{\lambda - t - iy}}\right) d\big(E_T(\lambda)h, h\big).$$

Obviously, one has

$$V_h(t+iy) \leqslant \int \frac{1}{((\lambda-t)^2 + y^2)^{1/4}}\, d\big(E_T(\lambda)h, h\big), \quad t \in \mathbb{R},\ y > 0,\ h \in \mathcal{H}.$$

Hence

$$V_h(t+iy)^p \leqslant \|h\|^{2(p-1)} \int \frac{1}{((\lambda-t)^2 + y^2)^{p/4}}\, d\big(E_T(\lambda)h, h\big), \quad p \in (1, \infty).$$

We show that for $p \in (1,2)$ and $-\infty < a < b < \infty$

$$C_p(h; a, b) := \sup_{y \in (0,1]} \int_a^b V_h(t+iy)^p\, dt < \infty.$$

Clearly,

$$\int_a^b V_h(t+iy)^p\,dt \leqslant \|h\|^{2(p-1)} \int_0^{\|T\|} d\big(E(\lambda)h,h\big) \int_a^b \frac{1}{((\lambda-t)^2+y^2)^{p/4}}\,dt$$
$$= \|h\|^{2(p-1)} \int_0^{\|T\|} d\big(E(\lambda)h,h\big) \int_{a-\lambda}^{b-\lambda} \frac{1}{(t^2+y^2)^{p/4}}\,dt.$$

Note, that for $p \in (1,2)$ and $-\infty < a < b < \infty$ the inequality

$$\int_{a-\lambda}^{b-\lambda} \frac{1}{(t^2+y^2)^{p/4}}\,dt \leqslant \int_{a-\|T\|}^{b} \frac{1}{t^{p/2}}\,dt =: \varkappa_p\big(b, a-\|T\|\big) < \infty,$$

holds. Hence $C_p(h;a,b) \leqslant \varkappa_p(b, a-\|T\|)\|h\|^{2p} < \infty$ for $p \in (1,2)$, $-\infty < a < b < \infty$ and $h \in \mathcal{H}$. By [10, Corollary 4.7], A_1 is purely absolutely continuous on any bounded interval (a,b). Thus, A_1 is purely absolutely continuous.

(iii) By [13, Proposition 5] A^K is defined by $A^K = A^* \restriction \ker(\varGamma_1 - M(0)\varGamma_0)$. It follows from (4.5) that $M(0) = -\sqrt{T}$. Therefore, A^K is defined by (4.9).

It follows from the extremal property of the Krein extension that $\ker(A^K) = \ker(A^*)$. Clearly, $f_h(x) := \exp(-x\sqrt{T})h \in L^2(\mathbb{R}_+, \mathcal{H})$, $h \in \operatorname{ran}(T^{1/4})$, since

$$\int_0^\infty \big\|\exp(-x\sqrt{T})h\big\|^2_{\mathcal{H}}\,dx = \int_0^{\|T\|} d\rho_h(t) \int_0^\infty e^{-2x\sqrt{t}}\,dx = \int_0^{\|T\|} \frac{1}{2\sqrt{t}}\,d\rho_h(t) < \infty,$$

where $\rho_h(t) := (E_T(t)h,h)$. Thus, $\mathfrak{H}_0' \subset \ker(A^*)$. It is easily seen that $\mathfrak{H}_0'$ is dense in $\mathfrak{H}_0$. To investigate the rest of the spectrum of A^K consider the Weyl function $M_K(\cdot)$ corresponding to A^K. It follows from (4.5) and (2.8) that

$$M_K(z) = M_{-\sqrt{T}}(z) = -\big(\sqrt{T} + M(z)\big)^{-1}$$
$$= -(\sqrt{T} + i\sqrt{z-T}\,)^{-1} = \frac{1}{z}(i\sqrt{z-T} - \sqrt{T}\,) = -\frac{2\sqrt{T}}{z} + \varPhi(z),$$

where $\varPhi(z) := \frac{1}{z}[i\sqrt{z-T} + \sqrt{T}\,]$. For $t > 0$ we get

$$\operatorname{Im} M_K(t+i0) = \operatorname{Im}\varPhi(t+i0) = t^{-1}\sqrt{t-T}\,E_T\big([0,t)\big). \tag{4.12}$$

Hence, by [10, Theorem 4.3], $\sigma_p(A^K) \cap (0,\infty) = \sigma_{sc}(A^K) \cap (0,\infty) = \emptyset$. It follows from (4.12) that $\operatorname{Im}(M_K(t+i0)) > 0$ for $t > t_0$. By Corollary 2.9 we find $\sigma_{ac}(A^K) = [t_0,\infty)$.

(iv) It follows from (4.7) and (4.12) that $d_M(t) = d_{M_K}(t) = \dim(\operatorname{ran}(E_T([0,t))))$ for $t > t_0$. Combining this equality with $\sigma_{ac}(A^K) = \sigma_{ac}(A^F) = [t_0,\infty)$, we conclude from Proposition 2.10(ii) that A^F and $(A^K)^{ac}$ are unitarily equivalent.

Passing to A_1, we assume that $1 \leqslant \dim(\operatorname{ran}(E_T([0,s)))) = p_1 < \infty$ for some $s > 0$. Let λ_k, $k \in \{1,\dots,p\}$, $p \leqslant p_1$, be the set of distinct eigenvalues within $[0,s)$. Since $M_0(t+iy)E_T([0,t))$ is the $p \times p$ matrix-function, the limit $M_0(t+i0)E_T([0,t))$ exists for $t \in [0,s) \setminus \bigcup_{k=1}^p \{\lambda_k\}$. It follows from (4.11) that

$$\operatorname{Im}\big(M_0(t)\big) = |T-t|^{-1/2}E_T\big([0,t)\big), \quad t \in [0,s) \setminus \bigcup_{k=1}^{p}\{\lambda_k\}.$$

This yields $d_{M_0(t)} := \dim(\mathrm{ran}(\mathrm{Im}(M_0(t)))) = \dim(\mathrm{ran}(E_T([0,t)))) = d_M(t)$ for a.e. $t \in [0,s) \setminus \bigcup_{k=1}^p \{\lambda_k\}$, that is, for a.e. $t \in [0,s)$.

If $\dim(E_T([t_0,s))) = \infty$, then there exists a point $s_0 \in (0,s)$, such that $\dim(E_T([0,s_0])) = \infty$ and $\dim(E_T([0,s))) < \infty$ for $s \in [0,s_0)$. For any $t \in (s_0,s)$ choosing $\tau \in (s_0,t)$ we note that $\dim(\mathrm{ran}(E_T([0,\tau)))) = \infty$. We set $\mathcal{H}_\tau := E_T([0,\tau))\mathcal{H}$ and $\mathcal{H}_\infty := E_T([\tau,\infty))\mathcal{H}$ as well as $T_\tau := TE_T([0,\tau))$ and $T_\infty := TE_T([\tau,\infty))$. Further, we choose Hilbert–Schmidt operators D_τ and D_∞ in $\mathcal{H}_\tau$ and $\mathcal{H}_\infty$, respectively, such that $\ker(D_\tau) = \ker(D_\tau^*) = \ker(D_\infty) = \ker(D_\infty^*) = \{0\}$. According to the decomposition $\mathcal{H} = \mathcal{H}_\tau \oplus \mathcal{H}_\infty$ we have $M_0 = M_\tau \oplus M_\infty$, $D = D_\tau \oplus D_\infty$ and $d_{M_0^D}(t) = d_{M_\tau^{D_\tau}}(t) + d_{M_\infty^{D_\infty}}(t)$ for a.e. $t \in [0,\infty)$. Hence $d_{M_0^D}(t) \geqslant d_{M_\tau^{D_\tau}}(t)$ for a.e. $t \in [0,\infty)$. Clearly, $M_\tau(t+iy) = i(t+iy-T_\tau)^{-1/2}$. If $t > \tau$, then $t \in \varrho(T_\tau)$ and $M(t) := \text{s-}\lim_{y\downarrow 0} M(t+i0)$ exists and

$$M_\tau(t) := \text{s-}\lim_{y\to 0} M_\tau(t+iy) = i(t-T_\tau)^{-1/2} E_T\bigl([0,\tau)\bigr).$$

Hence $d_{M_\tau^{D_\tau}}(t) = \dim(\mathrm{ran}(E_T([0,\tau)))) = \infty$ for $t > s_0$. Hence $d_{M_0^D}(t) = d_M(t) = \infty$ for a.e. $t > s_0$ which yields $d_{M_0^D}(t) = d_M(t)$ for a.e. $t \in [0,\infty)$. Using Proposition 2.10(ii) we obtain that A_0^{ac} and A_1^{ac} are unitarily equivalent which shows A_0 and A_1 are unitarily equivalent. $\square$

Remark 4.5. To be self-consistent we prove the statements on A^D, A^N and A^K in the framework of boundary triplets. At the same time, the absolute continuity of A^D and A^N, as well as their unitary equivalence can be proved simpler using their tensor structure (see Appendix A.2). However, the Krein realization A^K cannot be treated in such a manner since A^K has no tensor structure, i.e., does not admit separation of variables.

4.2. Ac-minimality and strict ac-minimality of some realizations

Next we describe the spectral properties of any self-adjoint realization of $\mathcal{A}$. In particular, we show that the Friedrichs extension A^F of A is *ac*-minimal, although A does not satisfy conditions of Theorem 3.17.

Theorem 4.6. *Let $T \geqslant 0$, $T = T^* \in [\mathcal{H}]$, and $t_1 := \inf \sigma_{\mathrm{ess}}(T)$. Let also A be the symmetric operator defined by* (4.1) *and $\widetilde{A} = \widetilde{A}^* \in \mathrm{Ext}_A$. Then*:

(i) *The ac-part $\widetilde{A}^{ac} E_{\widetilde{A}}([t_1,\infty))$ is unitarily equivalent to $A^D E_{A^D}([t_1,\infty))$.*
(ii) *Dirichlet, Neumann and Krein realizations are ac-minimal and $\sigma(A^D) = \sigma(A^N) = \sigma_{ac}(A^K) \subseteq \sigma_{ac}(\widetilde{A})$.*
(iii) *The ac-part $\widetilde{A}^{ac}$ is unitarily equivalent to A^D whenever either*

$$(\widetilde{A}-i)^{-1} - \bigl(A^D - i\bigr)^{-1} \in \mathfrak{S}_\infty(\mathfrak{H}) \quad \textit{or} \quad (\widetilde{A}-i)^{-1} - \bigl(A^K - i\bigr)^{-1} \in \mathfrak{S}_\infty(\mathfrak{H}).$$

Proof. By [38, Corollary 4.2] it suffices to consider realizations $\widetilde{A} = \widetilde{A}^*$ disjoint with A_0. According to Proposition 2.3(ii) such $\widetilde{A}$ admits a representation $\widetilde{A} = A_B$ with $B = B^* \in \mathcal{C}(\mathcal{H})$.

(i) Let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a boundary triplet for A^* defined by (4.4). In accordance with Proposition 2.10 we calculate $d_{M_B^K}(t)$ where $M_B(\cdot) := (B - M(\cdot))^{-1}$ is the generalized Weyl function of the extension A_B, cf. (2.8), and $K \in \mathfrak{S}_2(\mathcal{H})$ satisfying $\ker(K) = \ker(K^*) = \{0\}$. Clearly,

$$\mathrm{Im}\bigl(M_B(z)\bigr) = M_B(z)^* \,\mathrm{Im}\bigl(M(z)\bigr) M_B(z), \quad z \in \mathbb{C}_+. \tag{4.13}$$

Since $\mathrm{Re}(\sqrt{z-\lambda}) > 0$ for $z = t+iy$, $y > 0$, it follows from (4.5) that

$$\mathrm{Im}\bigl(M(z)\bigr) = \int_{[0,\infty)} \mathrm{Re}(\sqrt{z-\lambda})\, dE_T(\lambda) \geqslant \int_{[0,\tau)} \mathrm{Re}(\sqrt{z-\lambda})\, dE_T(\lambda). \tag{4.14}$$

It is easily seen that

$$\mathrm{Re}(\sqrt{z-\lambda}) \geqslant \sqrt{t-\lambda} \geqslant \sqrt{t-\tau}, \quad \lambda \in [0,\tau),\ t > \tau. \tag{4.15}$$

Combining (4.13) with (4.14) and (4.15) we get

$$\mathrm{Im}\big(M_B(t+iy)\big) \geqslant \sqrt{t-\tau}\, M_B(t+iy)^* E_T\big([0,\tau)\big) M_B(t+iy), \quad t > \tau > 0.$$

Let Q be a finite dimensional orthogonal projection, $Q \leqslant E_T([0,\tau))$. Hence

$$\mathrm{Im}\big(M_B(t+iy)\big) \geqslant \sqrt{t-\tau}\, M_B(t+iy)^* Q\, M_B(t+iy), \quad t > \tau > 0,\ y > 0.$$

Setting $\mathcal{H}_1 = \mathrm{ran}(Q)$, $\mathcal{H}_2 := \mathrm{ran}(Q^\perp)$, and choosing $K_2 \in \mathfrak{S}_2(\mathcal{H}_2)$ and satisfying $\ker(K_2) = \ker(K_2^*) = \{0\}$, we define a Hilbert–Schmidt operator $K := Q \oplus K_2 \in \mathfrak{S}_2(\mathcal{H})$. Clearly, $\ker(K) = \ker(K^*) = \{0\}$ and,

$$\mathrm{Im}\big(K^* M_B(t+iy)K\big) \geqslant \sqrt{t-\tau}\, K^* M_B(t+iy)^* Q\, M_B(t+iy)K, \quad t > \tau > 0. \tag{4.16}$$

Since $M_B(\cdot) \in R_{\mathcal{H}}$ and $Q, K \in \mathfrak{S}_2(\mathcal{H})$, the limits

$$K^* M_B(t)^* Q := \operatorname*{s-lim}_{y\downarrow 0} K^* M_B(t+iy)^* Q \quad \text{and}$$
$$(Q\, M_B K)(t) := \operatorname*{s-lim}_{y\downarrow 0} Q\, M_B(t+iy)K$$

exist for a.e. $t \in \mathbb{R}$ (see [6]). Therefore passing to the limit $y \to 0$ in (4.16), we arrive at the inequality

$$\mathrm{Im}\big(M_B^K(t)\big) \geqslant \sqrt{t-\tau}\big(K^* M_B(t)^* Q\big)\big(Q\, M_B K(t)\big), \quad t > \tau > 0,\ y > 0.$$

It follows that

$$\dim\big(\mathrm{ran}\big((Q\, M_B K)(t)\big)\big) \leqslant \dim\big(\mathrm{ran}\big(\mathrm{Im}\, M_B^K(t)\big)\big) = d_{M_B^K}(t), \quad t > \tau. \tag{4.17}$$

We set $\widetilde{M}_B^Q(z) := Q\, M_B(z) Q \upharpoonright \mathcal{H}_1$. Since $\dim(\mathcal{H}_1) < \infty$ the limit $\widetilde{M}_B^Q(t) := \text{s-lim}_{y\downarrow 0} \widetilde{M}_B^Q(t+iy)$ exists for a.e. $t \in \mathbb{R}$. Since $(Q\, M_B K)(t) \upharpoonright \mathcal{H}_1 = \widetilde{M}_B^Q(t)$, relation (4.17) yields the inequality

$$\dim\big(\mathrm{ran}\big(\widetilde{M}_B^Q(t)\big)\big) \leqslant \dim\big(\mathrm{ran}\big((Q\, M_B K)(t)\big)\big) \leqslant d_{M_B^K}(t) \tag{4.18}$$

for a.e. $t \in [\tau, \infty)$.

Because $\dim(\mathcal{H}_1) < \infty$ and $\ker(\widetilde{M}_B^Q(z)) = \{0\}$, $z \in \mathbb{C}$, we easily get by repeating the corresponding reasonings of the proof of Theorem 3.17 that $\mathrm{ran}(\widetilde{M}_B^Q(t)) = \mathcal{H}_1$ for a.e. $t \in \mathbb{R}$. Therefore (4.18) yields $\dim(\mathcal{H}_1) \leqslant d_{M_B^K}(t)$ for a.e. $t \in [\tau, \infty)$.

If $\tau > t_1$, then $\dim(E_T([0,\tau))\mathcal{H}) = \infty$ and the dimension of a projection $Q \leqslant E_T([0,\tau))$ can be arbitrary. Thus, $d_{M_B^K}(t) = \infty$ for a.e. $t > \tau$. Since $\tau > t_1$ is arbitrary we get $d_{M_B^K}(t) = \infty$ for a.e. $t > t_1$. By Proposition 2.10(ii) the operator $\widetilde{A}^{ac} E_{\widetilde{A}}([t_1,\infty))$ is unitarily equivalent to $A_0 E_{A_0}([t_1,\infty))$.

(ii) If $\tau \in (t_0, t_1)$, then $\dim(E_T([0,\tau))\mathcal{H}) =: p(\tau) < \infty$. Hence, $\dim(Q\mathcal{H}) \leqslant p(\tau)$ which shows that $d_{M_B^K}(t) \geqslant p(\tau)$ for a.e. $t \in (\tau, t_1)$. Since τ is arbitrary, we obtain $d_{M_B^K}(t) \geqslant p(\tau)$ for a.e. $t \in [0, t_1)$. Applying Proposition 2.10(i) we prove that A^D is ac-minimal. Now Theorem 4.4(iv) completes the proof of (ii).

(iii) By Lemma 4.3 the invariant maximal normal function $\mathfrak{m}^+(t)$ is finite for $t \in \mathbb{R}$. By Theorem 2.11 $\widetilde{A}^{ac}$ and $(A^F)^{ac}$ are unitarily equivalent. Similarly we prove that $\widetilde{A}^{ac}$ and $(A^K)^{ac}$ are unitarily equivalent. To complete the proof it remains to apply Theorem 4.4(i). $\square$

Next we investigate strict *ac*-minimality of realizations.

Corollary 4.7. *Let the assumptions of Theorem* 4.6 *be satisfied. Let also* $t_0 := \inf\sigma(T) = \inf\sigma_{\mathrm{ess}}(T) =: t_1$. *Then*:

(i) *Dirichlet, Neumann and Krein realizations are strictly ac-minimal*;
(ii) *The ac-part* $\widetilde{A}^{ac}$ *of* $\widetilde{A}$ *is unitarily equivalent to* A^D, *provided that*

$$(\widetilde{A} - i)^{-1} - \left(A^N - i\right)^{-1} \in \mathfrak{S}_\infty(\mathfrak{H}). \tag{4.19}$$

Proof. (i) This statement follows from Theorem 4.6(i) and Theorem 4.4.

(ii) By the Weyl theorem inclusion (4.19) yields $\sigma_{\mathrm{ess}}(\widetilde{A}) = \sigma_{\mathrm{ess}}(A^N)$. Since $\sigma_{\mathrm{ess}}(A^N) = \sigma_{ac}(A^N) = [t_0, \infty)$ we have $\sigma_{\mathrm{ess}}(\widetilde{A}) = [t_0, \infty)$. Combining this equality with Theorem 4.6(i) we get $[t_0, \infty) = \sigma_{\mathrm{ess}}(\widetilde{A}) \subseteq \sigma_{ac}(\widetilde{A})$. Thus, $\sigma_{ac}(\widetilde{A}) = [t_0, \infty)$ and $\widetilde{A}^{ac} = \widetilde{A}^{ac} E_{\widetilde{A}}([t_0, \infty))$. Combining Theorem 4.4(i) with Theorem 4.6(i) we get that $\widetilde{A}^{ac}$ is unitarily equivalent to A^D. $\square$

Remark 4.8. According to (4.10) the Weyl function $M_0(\cdot)$ of the Neumann extension A^N does not satisfy the condition $\mathfrak{m}^+(t) < \infty$, $t \in \mathbb{R}$ (cf. (2.11)). Therefore Corollary 4.7(ii) shows that the assumption $\mathfrak{m}^+(t) < \infty$ of Theorem 2.11 is not necessary for the validity of its conclusion.

Corollary 4.9. *Let the assumptions of Theorem* 4.6 *be satisfied. Then* A^D *is strictly ac-minimal if and only if* $t_0 = t_1$.

Proof. Let $t_0 < t_1$. Then there is a decomposition $T = T_{\mathrm{fin}} \oplus T_\infty$ such that T_{fin} acts in a finite dimensional Hilbert space $\mathcal{H}_{\mathrm{fin}}$ and $t_0 = \inf\sigma(T_{\mathrm{fin}})$ and $T_\infty = T_\infty^* \in \mathcal{C}(\mathcal{H}_\infty)$ and $t_0 < t_\infty := \inf\sigma(T_\infty) \leqslant t_1$. This leads to the decomposition $A = A_{\mathrm{fin}} \oplus A_\infty$ where A_{fin} and A_∞ are defined analogously to (4.1). Clearly $A^D = A^D_{\mathrm{fin}} \oplus A^D_\infty$. By Theorem 4.4 both extensions A^D_{fin} and A^D_∞ are absolutely continuous and their spectra are given by $\sigma(A^D_{\mathrm{fin}}) = [t_0, \infty)$ and $\sigma(A^D_\infty) = [t_\infty, \infty)$. Since $\dim(\mathcal{H}_\infty) = \infty$ the deficiency indices of A_∞ are infinite. We note that $(-\infty, t_\infty)$ is a spectral gap for A_∞. Using a result of Brasche [9] there exists an extension $\widetilde{A}_\infty = \widetilde{A}_\infty^* \in \mathrm{Ext}A_\infty$ such that $\sigma(\widetilde{A}_\infty) \subseteq [t_0, \infty)$, the part $\widetilde{A}_\infty E_{\widetilde{A}_\infty}([t_0, t_\infty))$ is absolutely continuous and $N_{\widetilde{A}_\infty^{ac}}(t) = \infty$ for $t \in [t_0, t_1)$.

Let $\widetilde{A} := A^D_{\mathrm{fin}} \oplus \widetilde{A}_\infty$. The operator $\widetilde{A}$ is a self-adjoint extension of A such that $\sigma(\widetilde{A}) = \sigma(A^D) = [t_0, \infty)$. The parts $A^D E_{A^D}([t_0, t_\infty))$ and $\widetilde{A} E_{\widetilde{A}}([t_0, t_\infty))$ are absolutely continuous. However, the absolutely continuous parts of both extensions are not unitarily equivalent. Indeed, for a.e. $t \in [t_0, t_\infty)$ one has $N_{A^D}(t) < \infty$ but $N_{\widetilde{A}^{ac}}(t) = \infty$, by construction. Hence A^D is not strictly *ac*-minimal that yields $t_0 = t_1$. The converse follows from Corollary 4.7(i). $\square$

5. Sturm–Liouville operators with unbounded operator potentials

5.1. Regularity properties

In this subsection we consider the differential expression (4.1) with unbounded non-negative $T = T^* (\in \mathcal{C}(\mathcal{H}))$ in $\mathfrak{H} := L^2(\mathbb{R}_+, \mathcal{H})$. The minimal operator $A := A_{\min} := \overline{\mathcal{A}}$, cf. (1.1) and (1.2), is densely defined and non-negative.

Let $\mathcal{H}_1(T)$ be the Hilbert space obtained by equipping $\mathrm{dom}(T)$ with the graph norm. Moreover, for any $s \geqslant 0$ we equip $\mathrm{dom}(T^s)$ with the graph norm

$$\|u\|_s = \left(\|u\|^2_{\mathcal{H}} + \left\|T^s u\right\|^2_{\mathcal{H}}\right)^{1/2} = \left\|\left(I + T^{2s}\right)^{1/2} u\right\|, \quad s \geqslant 0,\ u \in \mathrm{dom}\left(T^s\right), \tag{5.1}$$

and denote by $\mathcal{H}_s(T)$ the corresponding Hilbert space. Following [34, Definition I.2.1] we introduce the intermediate spaces $[X,Y]_\theta$, $\theta \in [0,1]$, between $X = \mathcal{H}_1(T)$ and $Y = \mathcal{H}_0(T) := \mathcal{H}$ by setting $[X,Y]_\theta := \mathcal{H}_{1-\theta}(T)$, $\theta \in [0,1]$.

Furthermore, by $\mathcal{H}_s(T)$, $s < 0$, we denote the completion of $\mathcal{H}$ with respect to the "negative" norm

$$\|u\|_s = \big\|\big(I + T^{-2s}\big)^{-1/2} u\big\|_{\mathcal{H}}, \quad s < 0,\ u \in \mathcal{H}. \tag{5.2}$$

At first, we describe the domain $\mathrm{dom}(A)$ of the minimal operator A. For this purpose, following [34] we introduce the Hilbert spaces $W^{2,2}_T(\mathbb{R}_+, \mathcal{H}) := W^{2,2}(\mathbb{R}_+, \mathcal{H}) \cap L^2(\mathbb{R}_+, \mathcal{H}_1(T))$, equipped with the Hilbert norms

$$\|f\|^2_{W^{2,2}_T} = \int_{\mathbb{R}_+} \big(\big\|f''(t)\big\|^2_{\mathcal{H}} + \big\|f(t)\big\|^2_{\mathcal{H}} + \big\|Tf(t)\big\|^2_{\mathcal{H}}\big)\,dt. \tag{5.3}$$

Obviously we have $\mathcal{D}_0 \subseteq W^{2,2}_T(\mathbb{R}_+, \mathcal{H})$ where $\mathcal{D}_0$ is given by (1.2). The closure of $\mathcal{D}_0$ in $W^{2,2}_T(\mathbb{R}_+, \mathcal{H})$ coincides with $W^{2,2}_{0,T}(\mathbb{R}_+, \mathcal{H}) := \{f \in W^{2,2}_T(\mathbb{R}_+, \mathcal{H}):\ f(0) = f'(0) = 0\}$ which yields $W^{2,2}_{0,T}(\mathbb{R}_+, \mathcal{H}) \subseteq \mathrm{dom}(A)$.

Lemma 5.1. *Let $T = T^*$ be a non-negative operator in $\mathcal{H}$. Then the domain* $\mathrm{dom}(A)$ *equipped with the graph norm coincides with the Hilbert space $W^{2,2}_{0,T}(\mathbb{R}_+, \mathcal{H})$ algebraically and topologically.*

Proof. Obviously, for any $f \in \mathcal{D}_0$ we have

$$\|\mathcal{A}f\|^2_{\mathfrak{H}} = \int_{\mathbb{R}_+} \big\|f''(x)\big\|^2_{\mathcal{H}}\,dx + \int_{\mathbb{R}_+} \big\|Tf(x)\big\|^2_{\mathcal{H}}\,dx - 2\,\mathrm{Re}\left\{\int_{\mathbb{R}_+} \big(f''(x), Tf(x)\big)_{\mathcal{H}}\,dx\right\}.$$

Integrating by parts we find

$$\int_{\mathbb{R}_+} \big(f''(x), Tf(x)\big)\,dx = -\int_{\mathbb{R}_+} \big\|\sqrt{T} f'(x)\big\|^2_{\mathcal{H}}\,dx, \quad f \in \mathcal{D}_0.$$

Hence for any $f \in \mathcal{D}_0$

$$\|\mathcal{A}f\|^2_{\mathfrak{H}} = \int_{\mathbb{R}_+} \big\|f''(x)\big\|^2\,dx + \int_{\mathbb{R}_+} \big\|Tf(x)\big\|^2\,dx + 2\int_{\mathbb{R}_+} \big\|\sqrt{T} f'(x)\big\|^2_{\mathcal{H}}\,dx.$$

Combining this relation with (5.3) yields

$$\|f\|^2_{W^{2,2}_T} \leqslant \|\mathcal{A}f\|^2_{\mathfrak{H}} + \|f\|^2_{\mathfrak{H}}, \quad f \in \mathcal{D}_0.$$

Furthermore, by the Schwartz inequality,

$$2\left|\mathrm{Re}\left\{\int_{\mathbb{R}_+} \big(f''(x), Tf(x)\big)_{\mathcal{H}}\,dx\right\}\right| \leqslant \|f\|^2_{W^{2,2}_T}, \quad f \in \mathcal{D}_0,$$

which gives

$$\|\mathcal{A}f\|^2_{\mathfrak{H}} + \|f\|^2 \leqslant 2\|f\|^2_{W^{2,2}_T}, \quad f \in \mathcal{D}_0.$$

Thus, we arrive at the two-sided estimate

$$\|f\|^2_{W^{2,2}_T} \leqslant \|\mathcal{A}f\|^2_{\mathfrak{H}} + \|f\|^2_{\mathfrak{H}} \leqslant 2\|f\|^2_{W^{2,2}_T}, \quad f \in \mathcal{D}_0.$$

Since $\mathcal{D}_0$ is dense in $W^{2,2}_{0,T}(\mathbb{R}_+, \mathcal{H})$ we obtain that $\mathrm{dom}(A)$ coincides with $W^{2,2}_{0,T}(\mathbb{R}_+, \mathcal{H})$ algebraically and topologically. $\square$

In opposite to the case of the minimal operator $A = A_{\min}$ the maximal operator $A_{\max} = A^*$ clearly satisfies $W^{2,2}_T(\mathbb{R}_+, \mathcal{H}) \subset \mathrm{dom}(A_{\max})$, though $\mathrm{dom}(A_{\max}) \neq W^{2,2}_T(\mathbb{R}_+, \mathcal{H})$ if T is not bounded. However, $W^{2,2}_T(\mathbb{R}_+, \mathcal{H})$ is dense in the Hilbert space $\mathfrak{H}_{+,A^*}$ obtained equipping $\mathrm{dom}(A_{\max})$ with the graph norm of A^*. It was shown for the first time in [19] (see also [20, Section 4.1]) that the trace mapping

$$\{\gamma_0, \gamma_1\} : W^{2,2}_T(I, \mathcal{H}) \to \mathcal{H}_{3/4}(T) \oplus \mathcal{H}_{1/4}(T), \quad \{\gamma_0, \gamma_1\}f = \{f(a), f'(a)\},$$

can be extended to a continuous (non-surjective) mapping

$$\{\gamma_0, \gamma_1\} : \mathrm{dom}(A_{\max}) \to \mathcal{H}_{-1/4}(T) \oplus \mathcal{H}_{-3/4}(T). \tag{5.4}$$

It is also shown in [20, Theorem 4.1.1] that $y(\cdot) \in \mathrm{dom}(A_{\max})$ if and only if the following conditions are satisfied:

(i) $y'(\cdot)$ exists and is an absolutely continuous function on I into $\mathcal{H}_{-1}(T)$;
(ii) $\mathcal{A}y \in L^2(I, \mathcal{H})$.

This result is similar to that for elliptic operators with smooth coefficients in domains with smooth boundary, cf. [24,33]. A similar statement holds also for the operator $A_{\max} = A^*_{\min}$ considered in $L^2(\mathbb{R}_+, \mathcal{H})$, cf. [13, Section 9]. This description of $\mathrm{dom}(A_{\max})$ together with relation (5.4) makes it possible to introduce the following extensions of A

$$\begin{aligned} \widehat{A}^D &= A^* \upharpoonright \{f \in \mathrm{dom}(A^*) : \gamma_0 f = 0\}, \\ \widehat{A}^N &= A^* \upharpoonright \{f \in \mathrm{dom}(A^*) : \gamma_1 f = 0\}, \end{aligned} \tag{5.5}$$

which turns out to be self-adjoint (see [20]).

Next, we investigate the Friedrichs extension A^F and the Krein extension A^K of the operator $A \geqslant 0$. We define the Neumann realization $\widetilde{A}^N$ as the self-adjoint operator associated with the closed quadratic form $\mathfrak{t}_N$,

$$\mathfrak{t}_N[f] := \int_0^\infty \{\|f'(x)\|^2_{\mathcal{H}} + \|\sqrt{T}f(x)\|^2_{\mathcal{H}}\}\,dx = \|f\|^2_{W^{1,2}_{\sqrt{T}}} - \|f\|^2_{L^2(\mathbb{R}_+,\mathcal{H})}, \tag{5.6}$$

$f \in \mathrm{dom}(\mathfrak{t}_N) := W^{1,2}_{\sqrt{T}}(\mathbb{R}_+, \mathcal{H})$. Clearly, $\widetilde{A}^N \in \mathrm{Ext}_A$. In the case of bounded T one has $\widetilde{A}^N = A_1$ where A_1 is defined in Theorem 4.4(ii).

We note that the closed quadratic form $\mathfrak{t}_F$ associated with Friedrich extensions A^F is given by $\mathfrak{t}_F := \mathfrak{t}_N \upharpoonright \mathrm{dom}(\mathfrak{t}_F)$,

$$\mathrm{dom}(\mathfrak{t}_F) := W^{1,2}_{0,\sqrt{T}}(\mathbb{R}_+, \mathcal{H}) := \{f \in W^{1,2}_{\sqrt{T}}(\mathbb{R}_+, \mathcal{H}) : f(0) = 0\}.$$

Proposition 5.2. *Let $T = T^* \in \mathcal{C}(\mathcal{H})$, $T \geqslant 0$, and $A := A_{\min}$. Let also $\mathcal{H}_n := \operatorname{ran}(E_T([n-1,n)))$, $T_n := TE_T([n-1,n))$, $n \in \mathbb{N}$, and let S_n be the closed minimal symmetric operator defined by* (4.1) *in $\mathfrak{H}_n := L^2(\mathbb{R}_+, \mathcal{H}_n)$ with T replaced by T_n. Then*:

(i) *The following decompositions hold*

$$A = \bigoplus_{n=1}^{\infty} S_n, \qquad A^F = \bigoplus_{n=1}^{\infty} S_n^F, \qquad A^K = \bigoplus_{n=1}^{\infty} S_n^K, \qquad \widetilde{A}^N = \bigoplus_{n=1}^{\infty} S_n^N. \tag{5.7}$$

(ii) *The domain* $\operatorname{dom}(A^F)$ *equipped with the graph norm coincides with a closed subspace $\{f \in W_T^{2,2}(\mathbb{R}_+, \mathcal{H})$: $f(0) = 0\}$ of $W_T^{2,2}(\mathbb{R}_+, \mathcal{H})$, algebraically and topologically. Moreover, $A^F = \widehat{A}^D = A^D = (A^D)^*$ where $\widehat{A}^D$ and A^D are defined by* (5.5) *and* (1.3)*, respectively. In particular, A^D is self-adjoint. Hence,*

$$\operatorname{dom}\bigl(A^F\bigr) := \bigl\{f \in \operatorname{dom}\bigl(A^*\bigr):\ \gamma_0 f = 0\bigr\} = \bigl\{f \in W_T^{2,2}(\mathbb{R}_+, \mathcal{H}):\ f(0) = 0\bigr\}.$$

(iii) *The domain* $\operatorname{dom}(\widetilde{A}^N)$ *equipped with the graph norm coincides with a closed subspace $\{f \in W_T^{2,2}(\mathbb{R}_+, \mathcal{H})$: $f'(0) = 0\}$ of $W_T^{2,2}(\mathbb{R}_+, \mathcal{H})$ algebraically and topologically. Moreover, $\widetilde{A}^N = \widehat{A}^N = A^N = (A^N)^*$, where $\widehat{A}^N$ and A^N are defined by* (5.5) *and* (1.3)*, respectively. In particular,*

$$\operatorname{dom}\bigl(\widetilde{A}^N\bigr) := \bigl\{f \in \operatorname{dom}\bigl(A^*\bigr):\ \gamma_1 f = 0\bigr\} = \bigl\{f \in W_T^{2,2}(\mathbb{R}_+, \mathcal{H}):\ f'(0) = 0\bigr\}.$$

Proof. (i) Equipping $\operatorname{dom}(S_n)$ with the graph norm we obtain the Hilbert space $\mathfrak{H}_{+,S_n}$ (cf. with (3.16)). Since T_n is bounded the Hilbert space $\mathfrak{H}_{+,S_n}$ coincides with $W_{T_n}^{2,2}(\mathbb{R}_+, \mathcal{H}_n)$, $n \in \mathbb{N}$, algebraically and topologically, cf. Lemma 5.1. Therefore the obvious identity

$$W_T^{2,2}(\mathbb{R}_+, \mathcal{H}) = \bigoplus_{n \in \mathbb{N}} W_{T_n}^{2,2}(\mathbb{R}_+, \mathcal{H}_n)$$

yields $\mathfrak{H}_{+,A} = \bigoplus_{n \in \mathbb{N}} \mathfrak{H}_{+,S_n}$. This relation proves the first identity of (5.7). The second and the third identities are implied by Corollary 3.10.

To prove the last relation in (5.7) we set $S^N := \bigoplus_{n=1}^{\infty} S_n^N$. Since $S_n^N = (S_n^N)^* \in \operatorname{Ext}_{S_n}$ and $A = \bigoplus_{n=1}^{\infty} S_n$, S^N is a self-adjoint extension of A, $S^N \in \operatorname{Ext}_A$. Let $f = \bigoplus_{n=1}^{\infty} f_n \in \mathfrak{H}$ where $\mathfrak{H} = \bigoplus_{n=1}^{\infty} \mathfrak{H}_n$. Denoting by $\mathfrak{t}'_N$ the quadratic form associated with S^N we find $f = \bigoplus_{n=1}^{\infty} f_n \in \operatorname{dom}(\mathfrak{t}'_N)$ if and only if $f_n \in \operatorname{dom}(\mathfrak{t}_n)$, $n \in \mathbb{N}$, and $\sum_{n=1}^{\infty} \mathfrak{t}_n[f_n] < \infty$ where $\mathfrak{t}_n$ is the quadratic form associated with S_n^N, $n \in \mathbb{N}$. If $f \in \operatorname{dom}(\mathfrak{t}'_N)$, then

$$\begin{aligned} \mathfrak{t}'_N[f] = \sum_{n=1}^{\infty} \mathfrak{t}_n[f_n] &= \sum_{n=1}^{\infty} \int_0^{\infty} \bigl\{\bigl\|f'_n(x)\bigr\|^2_{\mathcal{H}_n} + \bigl\|\sqrt{T}_n f_n(x)\bigr\|^2_{\mathcal{H}_n}\bigr\}\,dx \\ &= \int_0^{\infty} \bigl\{\bigl\|f'(x)\bigr\|^2_{\mathcal{H}} + \bigl\|\sqrt{T} f(x)\bigr\|^2_{\mathcal{H}}\bigr\}\,dx = \mathfrak{t}_N[f]. \end{aligned}$$

Hence $f \in \operatorname{dom}(\mathfrak{t}_N)$. Conversely, if $f = \bigoplus_1^{\infty} f_n \in \operatorname{dom}(\mathfrak{t}_N)$, then $f_n \in \operatorname{dom}(\mathfrak{t}_n)$, $n \in \mathbb{N}$, and $\sum_{n=1}^{\infty} \mathfrak{t}_n[f_n] < \infty$ which proves $f \in \operatorname{dom}(\mathfrak{t}'_N)$. Thus, $S^N = \widetilde{A}^N$.

(ii) Following the reasonings of Lemma 5.1 we find

$$\|f_n\|^2_{W_{T_n}^{2,2}} \leqslant \bigl\|S_n^F f_n\bigr\|^2_{\mathfrak{H}_n} + \|f_n\|^2_{\mathfrak{H}_n} \leqslant 2\|f_n\|^2_{W_{T_n}^{2,2}}, \quad n \in \mathbb{N}, \tag{5.8}$$

where $f_n \in \mathrm{dom}(S_n^F) = \{g_n \in W^{2,2}(\mathbb{R}_+, \mathcal{H}_n)\colon\ g_n(0) = 0\}$. Using representation (5.7) for A^F and setting $f^m := \bigoplus_{n=1}^m f_n$, $f_n \in \mathrm{dom}(F_n)$, we obtain from (5.8) that

$$\|f^m\|^2_{W_T^{2,2}} \leqslant \|A^F f^m\|^2_{\mathfrak{H}} + \|f^m\|^2_{\mathfrak{H}} \leqslant 2\|f^m\|^2_{W_T^{2,2}}, \quad m \in \mathbb{N}. \tag{5.9}$$

Since the set $\{f^m = \bigoplus_{n=1}^m f_n\colon\ f_n \in \mathrm{dom}(S_n^F),\ m \in \mathbb{N}\}$, is a core for A^F, inequality (5.9) remains valid for $f \in \mathrm{dom}(A^F)$. This shows that $\mathrm{dom}(A^F) = \{f \in W_T^{2,2}(\mathbb{R}_+, \mathcal{H})\colon\ f(0) = 0\} = \mathrm{dom}(A^D)$. Moreover, due to (5.9) the graph norm on $\mathrm{dom}(A^F)$ and the norm $\|\cdot\|_{W_T^{2,2}}$ restricted to $\mathrm{dom}(A^F)$ are equivalent.

(iii) Similarly to (5.8) one gets

$$\|f_n\|^2_{W_{T_n}^{2,2}} \leqslant \|S_n^N f_n\|^2_{\mathfrak{H}_n} + \|f_n\|^2 \leqslant 2\|f_n\|^2_{W_{T_n}^{2,2}}$$

for $f_n \in \mathrm{dom}(S_n^N) = \{g_n \in W^{2,2}(\mathbb{R}_+, \mathcal{H}_n)\colon\ g_n'(0) = 0\}$, $n \in \mathbb{N}$. The rest of the proof is similar to that of (ii). □

To state the next result we denote by $C_b(\mathbb{R}_+, \mathcal{H}_s)$, $s \in [0,1]$, the space of bounded continuous functions $f : \mathbb{R}_+ \to \mathcal{H}_s$. Moreover, for any $f \in W^{2,2}(\mathbb{R}_+, \mathcal{H})$ we put $\partial f := f'$ where the derivative is understood in the distributional sense.

Corollary 5.3. *Let the assumptions of Proposition* 5.2 *be satisfied and* $f \in \mathrm{dom}(A^D) \cup \mathrm{dom}(A^N)$. *Then*:

(i) $\partial f := f' \in L^2(\mathbb{R}_+, \mathcal{H}_{1/2}(T))$ *and the mappings*

$$\partial : \mathrm{dom}\big(A^D\big) \ni f \to f' \in L^2\big(\mathbb{R}_+, \mathcal{H}_{1/2}(T)\big),$$
$$\partial : \mathrm{dom}\big(A^N\big) \ni f \to f' \in L^2\big(\mathbb{R}_+, \mathcal{H}_{1/2}(T)\big)$$

are continuous;

(ii) $f(\cdot) \in C_b(\mathbb{R}_+, \mathcal{H}_{3/4}(T))$, $f'(\cdot) \in C_b(R_+, \mathcal{H}_{1/2}(T))$ *and the mappings*

$$\partial^j : \mathrm{dom}\big(A^D\big) \ni f \to f^{(j)} \in C_b\big(\mathbb{R}_+, \mathcal{H}_{3/4-j/2}(T)\big),$$
$$\partial^j : \mathrm{dom}\big(A^N\big) \ni f \to f^{(j)} \in C_b\big(\mathbb{R}_+, \mathcal{H}_{3/4-j/2}(T)\big),$$

$j \in \{0,1\}$, *are continuous. In particular,* $f(0) \in \mathcal{H}_{3/4}(T)$ *and* $f'(0) \in \mathcal{H}_{1/4}(T)$.

Proof. (i) It follows from Proposition 5.2(ii), (iii) that $u \in L^2(\mathbb{R}_+, X)$, $X = \mathcal{H}_1(T)$. Applying theorem on intermediate derivatives [34, Theorem I.2.3] to $X \subseteq Y = \mathcal{H}_0 := \mathcal{H}$, we obtain $f' \in L^2(\mathbb{R}_+, [X,Y]_{1/2}) = L^2(\mathbb{R}_+, \mathcal{H}_{1/2}(T))$. Moreover, by the same theorem, the mapping ∂ is continuous.

(ii) Combining Proposition 5.2(ii), (iii) with the trace theorem [34, Theorem 1.3.1] one proves (ii). □

Corollary 5.4. *Let* $\mathfrak{H}_{+,A^*}$ *be the Hilbert space equipping* $\mathrm{dom}(A^*)$ *with the graph norm. Then* $W_T^{2,2}(\mathbb{R}_+, \mathcal{H})$ *is dense in* $\mathfrak{H}_{+,A^*}$, *although* $W_T^{2,2}(\mathbb{R}_+, \mathcal{H}) \neq \mathrm{dom}(A^*)$.

Proof. By Proposition 5.2, $\mathrm{dom}(A^F) = \{f \in W_T^{2,2}(\mathbb{R}_+, \mathcal{H})\colon\ f(0) = 0\}$ and $\mathrm{dom}(A^N) = \{f \in W_T^{2,2}(\mathbb{R}_+, \mathcal{H})\colon\ f'(0) = 0\}$ algebraically and topologically. Therefore, by [34, Theorem 1.3.2.], it holds

$$\mathrm{dom}\big(A^F\big) + \mathrm{dom}\big(A^N\big) = W_T^{2,2}(\mathbb{R}_+, \mathcal{H}),$$
$$\mathrm{dom}\big(A^F\big) \cap \mathrm{dom}\big(A^N\big) = \mathrm{dom}(A) = W_{0,T}^{2,2}(\mathbb{R}_+, \mathcal{H}). \tag{5.10}$$

On the other hand, it follows from Theorem 5.7(ii) and Proposition 2.3 that the extensions A^D and A^N are disjoint but not transversal. Combining this fact with (5.10) we get that $W^{2,2}_T(\mathbb{R}_+, \mathcal{H})$ is dense in $\mathfrak{H}_{+,A^*}$. $\square$

Remark 5.5. (i) Lemma 5.1, Proposition 5.2 and Corollary 5.3 also hold for realizations of the differential expression $\mathcal{A}$ considered on a finite interval I, i.e., in the space $L^2(I, \mathcal{H})$. For this case Corollary 5.3 has originally been proved by M.L. Gorbachuk [19] (see also [20, Corollary 4.1.5 and Theorem 4.2.4]), by applying another method. We emphasize, however, that Lemma 5.1 and Proposition 5.2 are new even for the case of finite interval realizations.

Note also that Lemma 5.1 and Proposition 5.2 are similar to the classical regularity results for smooth elliptic differential expressions in domains with smooth boundary (see [4,24,34]).

(ii) At first glance the operators A^D and A^N defined by (1.3) are not necessarily closed realizations of $\mathcal{A}$ and their closures coincide with the realizations $\widehat{A}^D$ and $\widehat{A}^N$ defined by (5.5). It is interesting to note that by Corollary 5.4 the angle between (closed) subspaces $\mathrm{dom}(A^F)$ and $\mathrm{dom}(A^N)$ in $\mathfrak{H}_{+,A^*}$ is zero.

5.2. Operators on the semi-axis: Spectral properties

To extend Theorem 4.4 to the case of expression (1.1) with unbounded $T = T^* \geqslant 0$, we first construct a boundary triplet for A^*, applying Theorem 3.7 to the representation (5.7) for A.

Lemma 5.6. *Let the assumptions of Proposition 5.2 be satisfied. Then there is a sequence of boundary triplets $\widehat{\Pi}_n = \{\mathcal{H}_n, \widehat{\Gamma}_{0n}, \widehat{\Gamma}_{1n}\}$ for S^*_n such that $\widehat{\Pi} := \bigoplus_{n=1}^{\infty} \widehat{\Pi}_n =: \{\mathcal{H}, \widehat{\Gamma}_0, \widehat{\Gamma}_1\}$ forms a boundary triplet for A^*. Moreover, $A^F = A^* \upharpoonright \ker(\widehat{\Gamma}_0)$ and the corresponding Weyl function is given by*

$$\widehat{M}(z) = \frac{i\sqrt{z-T} + \mathrm{Im}(\sqrt{i-T}\,)}{\mathrm{Re}(\sqrt{i-T}\,)}, \quad z \in \mathbb{C}_+. \tag{5.11}$$

Proof. For any $n \in \mathbb{N}$ we choose a boundary triplet $\Pi_n = \{\mathcal{H}_n, \Gamma_{0n}, \Gamma_{1n}\}$ for S^*_n with Γ_{0n}, Γ_{1n} defined by (4.4). By Theorem 4.4(i) $S^F_n = S_{0n} = S^*_n \upharpoonright \ker(\Gamma_{0n})$ and by Lemma 4.2 the corresponding Weyl function is $M_n(z) = i\sqrt{z-T_n}$.

Following Theorem 3.7, i.e. using construction (3.22), we define a sequence of regularized boundary triplets $\widehat{\Pi}_n = \{\mathcal{H}_n, \widehat{\Gamma}_{0n}, \widehat{\Gamma}_{1n}\}$ for S^*_n by setting $R_n := (\mathrm{Re}(\sqrt{i-T_n}\,))^{1/2}$, $Q_n := -\mathrm{Im}(\sqrt{i-T_n}\,)$ and

$$\widehat{\Gamma}_{0n} := R_n \Gamma_{0n}, \qquad \widehat{\Gamma}_{1n} := R_n^{-1}(\Gamma_{1n} - Q_n \Gamma_{0n}), \quad n \in \mathbb{N}. \tag{5.12}$$

Hence $S^F_n = S_{0n}$ and the corresponding Weyl function $\widehat{M}_n(\cdot)$ is given by

$$\widehat{M}_n(z) = \frac{i\sqrt{z-T_n} + \mathrm{Im}(\sqrt{i-T_n}\,)}{\mathrm{Re}(\sqrt{i-T_n}\,)}, \quad z \in \mathbb{C}_+,\ n \in \mathbb{N}. \tag{5.13}$$

By Theorem 3.7, the direct sum $\widehat{\Pi} := \bigoplus_{n=1}^{\infty} \widehat{\Pi}_n = \{\mathcal{H}, \widehat{\Gamma}_0, \widehat{\Gamma}_1\}$ forms a boundary triplet for A^* and the corresponding Weyl function is

$$\widehat{M}(z) = \bigoplus_{n \in \mathbb{N}} \widehat{M}_n(z), \quad z \in \mathbb{C}_+. \tag{5.14}$$

Combining (5.14) with (5.13) we get (5.11). From Theorem 3.7 (cf. (3.4)) and Corollary 3.10 we get

$$A_0 = A^* \upharpoonright \ker(\widehat{\Gamma}_0) = \bigoplus_{n=1}^{\infty} S^*_n \upharpoonright \ker(\widehat{\Gamma}_{0n}) = \bigoplus_{n=1}^{\infty} S_{0n} = \bigoplus_{n=1}^{\infty} S^F_n = A^F \tag{5.15}$$

which proves the second assertion. $\square$

Next we extend Theorem 4.4 to the case of unbounded operator potentials.

Theorem 5.7. *Let $T = T^* \geqslant 0$, $t_0 := \inf \sigma(T)$. Let $A := A_{\min}$ be the minimal operator associated with $\mathcal{A}$, cf.* (1.1) *and let $\widehat{\Pi} = \{\mathcal{H}, \widehat{\Gamma}_0, \widehat{\Gamma}_1\}$ be the boundary triplet for A^* defined by Lemma* 5.6. *Then the following hold*:

(i) *The Dirichlet realization A^D, $\operatorname{dom}(A^D) := \{g \in W_T^{2,2}(\mathbb{R}_+, \mathcal{H}):\ g(0) = 0\}$ coincides with $A_0 := A^* \upharpoonright \ker(\widehat{\Gamma}_0) = A^F$, the Friedrichs extension. Moreover, A^D is absolutely continuous and $\sigma(A^D) = \sigma_{ac}(A^D) = [t_0, \infty)$.*

(ii) *The Neumann realization A^N, $\operatorname{dom}(A^N) := \{g \in W_T^{2,2}(\mathbb{R}_+, \mathcal{H}):\ g'(0) = 0\}$ is given by $A_{B^N} := A^* \upharpoonright \ker(\widehat{\Gamma}_1 - B^N \widehat{\Gamma}_0)$ where $B^N := T + \sqrt{I + T^2}$. Moreover, A^N is absolutely continuous $\sigma(A^N) = \sigma_{ac}(A^N) = [t_0, \infty)$.*

(iii) *The Krein extension A^K is given by $A_{B^K} := A^* \upharpoonright \ker(\widehat{\Gamma}_1 - B^K \widehat{\Gamma}_0)$, where*

$$B^K = \frac{1}{\sqrt{2}\sqrt{T} + \sqrt{T + \sqrt{1 + T^2}}} \frac{1}{\sqrt{T + \sqrt{1 + T^2}}}. \tag{5.16}$$

Moreover, it holds $\ker(A^K) = \mathfrak{H}_0 := \overline{\{e^{-x\sqrt{T}}h:\ h \in \operatorname{ran}(T^{1/4})\}}$, the restriction $A^K \upharpoonright \operatorname{dom}(A^K) \cap \mathfrak{H}_0^{\perp}$ is absolutely continuous and $A^K = 0_{\mathfrak{H}_0} \oplus (A^K)^{ac}$. In particular, $\sigma(A^K) = \{0\} \cup \sigma_{ac}(A^K)$ and $\sigma_{ac}(A^K) = [t_0, \infty)$.

(iv) *The realizations A^D, A^N and $(A^K)^{ac}$ are unitarily equivalent.*

Proof. (i) By Proposition 5.2(ii), $A^D = A^F$. By Lemma 5.6, $A^F = A_0$. Finally, combining Proposition 5.2(i) with Theorem 4.4(i) we get the last statement.

(ii) It is easily seen that with respect to the boundary triplet $\widehat{\Pi}_n = \{\mathcal{H}_n, \widehat{\Gamma}_{0n}, \widehat{\Gamma}_{1n}\}$ defined by (5.12), the extension S_n^N is given by $S_n^N = S_{B_n}$, where $B_n := T_n + \sqrt{1 + T_n^2}$, $n \in \mathbb{N}$. By Proposition 5.2(i), $A^N = \bigoplus_{n=1}^{\infty} S_n^N = \bigoplus_{n \in \mathbb{N}} S_{B_n}$. Using $\widehat{\Pi} = \bigoplus_{n \in \mathbb{N}} \widehat{\Pi}_n$ and setting $B^N = \bigoplus_{n \in \mathbb{N}} B_n$, we find $A^N = A_{B^N}$. The remaining part of the statement follows from Theorem 4.4(ii).

(iii) Using the polar decomposition $i - \lambda = \sqrt{1 + \lambda^2} e^{i\theta(\lambda)}$ where $\theta(\lambda) = \pi - \arctan(1/\lambda)$, $\lambda \geqslant 0$, we get

$$\operatorname{Re}(\sqrt{i - T}\,) = \int_0^\infty \sqrt[4]{1 + \lambda^2} \cos\bigl(\theta(\lambda)/2\bigr)\, dE_T(\lambda). \tag{5.17}$$

Setting $\varphi(\lambda) = \arctan(1/\lambda)$, $\lambda \geqslant 0$ and noting that $\cos(\varphi(\lambda)) = \lambda(1 + \lambda^2)^{-1/2}$, we find $\cos(\theta(\lambda)/2) = 2^{-1/2}(1 + \lambda^2)^{-1/4}(\lambda + \sqrt{1 + \lambda^2}\,)^{-1/2}$. Inserting this expression into (5.17) yields

$$\operatorname{Re}(\sqrt{i - T}\,) = 2^{-1/2}\bigl(T + \sqrt{1 + T^2}\,\bigr)^{-1/2}. \tag{5.18}$$

Similarly, taking into account $\sin(\theta(\lambda)/2) = \cos(\varphi(\lambda)/2)$ and $\cos(\varphi(\lambda)/2) = 2^{-1/2}(1 + \lambda^2)^{-1/4}(\lambda + \sqrt{1 + \lambda^2}\,)^{1/2}$, we get

$$\operatorname{Im}(\sqrt{i - T}\,) = \int_0^\infty \sqrt[4]{1 + \lambda^2} \cos\bigl(\varphi(\lambda)/2\bigr)\, dE_T(\lambda) = \frac{1}{\sqrt{2}} \sqrt{T + \sqrt{1 + T^2}}. \tag{5.19}$$

It follows from (5.11) with account of (5.18) and (5.19) that $M(0) := \text{s-lim}_{x \downarrow 0} M(-x) = B^K$, where B^K is defined by (5.16). At the same time, by [13, Proposition 5(iv)], the Krein extension A^K is given by the extension $A^K := A^* \upharpoonright \ker(\Gamma_1 - M(0)\Gamma_0)$. This yields the first statement. The remaining part is implied by Proposition 5.2(i) and Theorem 4.4(iii).

(iv) The assertion follows from Theorem 4.4(iv) and (5.7). $\square$

Next we extend Theorem 4.6 to the case of unbounded $T \geqslant 0$.

Theorem 5.8. *Let $T = T^* \geqslant 0$ and $t_1 := \inf \sigma_{\mathrm{ess}}(T)$. Let also $A = A_{\min}$ and $\widetilde{A} = \widetilde{A}^* \in \mathrm{Ext}_A$. Then the following hold*:

(i) *The absolutely continuous part $\widetilde{A}^{ac} E_{\widetilde{A}}([t_1,\infty))$ of $\widetilde{A}$ is unitarily equivalent to the part $A^D E_{A^D}([t_1,\infty))$ of A^D.*

(ii) *The Dirichlet, Neumann and Krein realizations are ac-minimal and $\sigma(A^D) = \sigma(A^N) = \sigma_{ac}(A^K) \subseteq \sigma_{ac}(\widetilde{A})$.*

(iii) *The ac-part $\widetilde{A}^{ac}$ is unitarily equivalent to A^D if either*

$$(\widetilde{A} - i)^{-1} - \left(A^F - i\right)^{-1} \in \mathfrak{S}_\infty(\mathfrak{H}) \quad \textit{or} \quad (\widetilde{A} - i)^{-1} - \left(A^K - i\right)^{-1} \in \mathfrak{S}_\infty(\mathfrak{H}). \tag{5.20}$$

Proof. Let $\widehat{\Pi} = \{\mathcal{H}, \widehat{\Gamma}_0, \widehat{\Gamma}_1\}$ be the boundary triplet defined in Lemma 5.6. By [38, Corollary 4.2], it suffices to assume in addition that $\widetilde{A}$ is disjoint with A_0, i.e., due to Proposition 2.3(ii), it admits a representation $\widetilde{A} = A_B$, $\mathrm{dom}(A_B) = \ker(\widehat{\Gamma}_1 - B\widehat{\Gamma}_0)$, with the boundary operator $B \in \mathcal{C}(\mathcal{H})$.

(i) In accordance with (2.8) the Weyl function corresponding to A_B is given by $\widehat{M}_B(z) = (B - \widehat{M}(z))^{-1}$, $z \in \mathbb{C}_+$, where $\widehat{M}(z)$ is the Weyl function corresponding to the triplet $\widehat{\Pi} = \{\mathcal{H}, \widehat{\Gamma}_0, \widehat{\Gamma}_1\}$ and defined by (5.11). Clearly,

$$\mathrm{Im}\big(\widehat{M}_B(z)\big) = \widehat{M}_B(z)^* \,\mathrm{Im}\big(\widehat{M}(z)\big)\widehat{M}_B(z), \quad z \in \mathbb{C}_+. \tag{5.21}$$

It follows from (5.18) that $(\mathrm{Re}(\sqrt{i-T}\,))^{-1} \geqslant \sqrt{2}$. Therefore (5.11) yields

$$\mathrm{Im}\big(\widehat{M}(z)\big) \geqslant \sqrt{2}\,\mathrm{Im}\big(M(z)\big) = \sqrt{2}\,\mathrm{Im}(i\sqrt{z-T}\,), \quad z \in \mathbb{C}_+, \tag{5.22}$$

cf. (4.5). Repeating the reasoning of the proof of Theorem 4.6(i), we obtain from (5.22) that $d_{\widehat{M}^D}(t) = \infty$ for a.e. $t \in [t_1,\infty)$, where $D = D^* \in \mathfrak{S}_2(\mathcal{H})$ and $\ker D = \{0\}$. Moreover, it follows from (5.21) that $d_{\widehat{M}_B^D}(t) = d_{\widehat{M}^D}(t) = \infty$ for a.e. $t \in [t_1,\infty)$. One completes the proof by applying Proposition 2.10.

(ii) To prove ac-minimality of A^D we follow the proof of Theorem 4.6(ii) applying estimates (5.22). The statement for A^D then follows from Proposition 2.10. The ac-minimality of A^N and A^K is then implied by Theorem 5.7(iv).

(iii) The Weyl function $\widehat{M}(\cdot)$ corresponding to $\widehat{\Pi}$ admits the representation (5.13) and (5.14). Let $\mathfrak{m}_n^+$ be the invariant maximal normal function defined by (2.11) with respect to $M_n(z) = i\sqrt{z-T}$. It follows from (4.6) that $\sup_{n\in\mathbb{N}} \mathfrak{m}_n^+(t) < \infty$ for $t \in \mathbb{R}$ because the estimate (4.6) does not depend on T_n. Therefore the first statement immediately follows from Theorem 2.11.

To prove the second statement we note that the operator B^K defined by (5.16) is bounded. Therefore, according to (2.8) the Weyl function of the operator A_{B^K} is

$$\widehat{M}_{B^K}(z) = \left(B^K - \widehat{M}(z)\right)^{-1}, \quad z \in \mathbb{C}_+.$$

Inserting expression (5.16) into this formula we get

$$\widehat{M}_{B^K}(z) = -\frac{1}{\sqrt{2}}\frac{1}{\sqrt{T} + i\sqrt{z-T}}\frac{1}{\sqrt{T+\sqrt{1+T^2}}} = \frac{1}{z\sqrt{2}}\frac{\sqrt{T} - i\sqrt{z-T}}{\sqrt{T+\sqrt{1+T^2}}}.$$

It follows that the strong limit $\widehat{M}_{B^K}(t+i0)$ exists for any $t \in \mathbb{R}\setminus\{0\}$ and

$$\widehat{M}_{B^K}(t) := \widehat{M}_{B^K}(t+i0) := \operatorname*{s-lim}_{y\to *0} M_{B^K}(t+iy) = -\frac{1}{t\sqrt{2}}\frac{\sqrt{T} - i\sqrt{t-T}}{\sqrt{T+\sqrt{1+T^2}}}.$$

Clearly, $\widehat{M}_{B^K}(t_0) \in [\mathcal{H}]$ for any $t_0 \in \mathbb{R} \setminus \{0\}$. By Theorem 2.11, the *ac*-parts of $\widetilde{A}$ and A^K are unitarily equivalent whenever $(\widetilde{A} - i)^{-1} - (A^K - i)^{-1} \in \mathfrak{S}_\infty(\mathfrak{H})$. This completes the proof. □

Finally, we generalize Corollary 4.7 to the case of unbounded operator potentials and complete Theorem 5.8 as well.

Corollary 5.9. *Let the assumptions of Theorem* 5.8 *be satisfied. Assume, in addition, that* $t_0 := \inf \sigma(T) = \inf \sigma_{\mathrm{ess}}(T) =: t_1$. *Then*:

(i) *The Dirichlet, Neumann and Krein realizations are strictly ac-minimal.*
(ii) *The ac-part* $\widetilde{A}^{ac}$ *of* $\widetilde{A}$ *is unitarily equivalent to* A^D *whenever* $(\widetilde{A} - i)^{-1} - (A^N - i)^{-1} \in \mathfrak{S}_\infty(\mathfrak{H})$.

Proof. (i) Since $t_0 = t_1$, it follows from Theorem 5.8(i), and Theorem 5.7(i) that for any $\widetilde{A} = \widetilde{A}^* \in \mathrm{Ext}_A$ the *ac*-part $\widetilde{A}^{ac} E_{\widetilde{A}}([t_0,\infty))$ of $\widetilde{A} E_{\widetilde{A}}([t_0,\infty))$ is unitarily equivalent to A^D, that is, A^D is strictly *ac*-minimal. Now the strict *ac*-minimality of A^N and A^K is implied by Theorem 5.7(iv).

(ii) The statement is proved in just the same way as Corollary 4.7(ii). □

Remark 5.10. Clearly, $t_0 \neq t_1$ if $T \geqslant 0$ and $(T + I)^{-1} \in \mathfrak{S}_\infty(\mathcal{H})$. Thus, in this case A^D is *ac*-minimal but not strictly *ac*-minimal.

Next we complete both Theorem 5.8 and Corollary 5.9 by showing that in contrast to the realizations admitting separation of variables (see Appendix A), the other realizations might have non-trivial non-negative singular spectrum.

Corollary 5.11. *Assume the conditions of Theorem* 5.8. *Assume also that* $t_0 = t_1$. *Then there exist self-adjoint extensions* $\widetilde{A}$ *of* A *with non-trivial non-negative singular spectrum, i.e.* $\sigma_s(\widetilde{A}) \cap \mathbb{R}_+ \neq \emptyset$ *and* $\sigma_{ac}(\widetilde{A}) \supset \mathbb{R}_+$.

Proof. Assume for definiteness that $t_0 = t_1 = 0$ and set $\mathcal{H}_1 := E_T([0,1))\mathcal{H}$ and $\mathcal{H}_2 := E_T([1,\infty))$. Since $t_1 = 0$ and T is unbounded, $\dim(\mathcal{H}_1) = \dim(\mathcal{H}_2) = \infty$. We set $T_1 := T \upharpoonright \mathcal{H}_1$ and $T_2 := T \upharpoonright \mathcal{H}_2$. Clearly, $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ and

$$A = A_1 \oplus A_2, \quad A_j = A_{j,\min} := -\frac{d^2}{dt^2} + T_j, \ j \in \{1,2\}.$$

Clearly, both symmetric operators A_1 and A_2 have infinite deficiency indices, $n_\pm(A_j) = \dim(\mathcal{H}_j) = \infty$, $j \in \{1,2\}$. Moreover, since $A_2 \geqslant I$, the interval $(0,1)$ is the gap of A_2. By [9] there is an extension $\widetilde{A}_2 = \widetilde{A}_2^*$ such that the part $\widetilde{A}_2 E_{\widetilde{A}}([0,1])$ is purely singular. Choosing any extension $\widetilde{A}_1 = \widetilde{A}_1^*$ of A_1 and setting $\widetilde{A} := \widetilde{A}_1 \oplus \widetilde{A}_2$ we get a self-adjoint extension of A such that $\widetilde{A}^s E_{\widetilde{A}}([0,1]) \neq 0$. At the same time, by Theorem 5.8(i), the part $\widetilde{A}^{ac} E_{\widetilde{A}}([0,\infty))$ is unitarily equivalent to A^D. Hence $\sigma_{ac}(\widetilde{A}) \supset \mathbb{R}_+$. □

Next we apply Theorem 5.8 to realizations $\widehat{A}^C$ of the form $\widehat{A}^C = A^* \upharpoonright \mathrm{dom}(\widehat{A}^C)$ where $C = C^* \in \mathcal{C}(\mathcal{H})$ and

$$\mathrm{dom}\big(\widehat{A}^C\big) := \big\{ f \in \mathrm{dom}\big(A^*\big) \colon \gamma_0 f \in \mathrm{dom}(C), \ \gamma_1 f \in \mathcal{H}, \ \gamma_1 f = C\gamma_0 f \big\}.$$

It is shown in [21,22] (see also [13, Section 9]) that the operator $\widehat{A}^C$ is self-adjoint provided that $C = C^*$ is strongly subordinated to $T^{1/2}$, i.e.,

$$\|Cf\| \leqslant a \big\| T^{1/2} f \big\| + b\|f\|, \quad f \in \mathrm{dom}\big(T^{1/2}\big), \ 0 < a < 1, \ b > 0.$$

Corollary 5.12. *Let the assumptions of Theorem* 5.8 *be satisfied. Assume also that $\widehat{A}^C$ is self-adjoint, $\widehat{A}^C = (\widehat{A}^C)^*$. If either*

$$(T+I)^{-1/2}C(T+I)^{-1/2} \in \mathfrak{S}_\infty(\mathcal{H}) \quad or \quad (T+I)^{-1} \in \mathfrak{S}_\infty(\mathcal{H}), \tag{5.23}$$

then the ac-part $(\widehat{A}^C)^{ac}$ of $\widehat{A}^C$ is unitarily equivalent to A^D.

Proof. According to [21,22] $(\widehat{A}^C - i)^{-1} - (A^N - i)^{-1} \in \mathfrak{S}_p$ provided that either $(T+I)^{-1/2}C(T+I)^{-1/2} \in \mathfrak{S}_p(\mathcal{H})$ or $(T+I)^{-1} \in \mathfrak{S}_p(\mathcal{H})$ for $p \in [1,\infty]$. It remains to apply Theorem 5.8(iii). □

6. Applications

In this section we apply previous results on Schrödinger operators in the half-space. To this end we denote by $L = L_{\min}$ the minimal elliptic operator associated with the differential expression

$$\mathcal{L} := -\frac{\partial^2}{\partial t^2} - \sum_{j=1}^{n} \frac{\partial^2}{\partial x^2} + q(x), \quad q = \bar{q} \in L^\infty(\mathbb{R}^n),$$

in $\mathfrak{H} = L^2(\mathbb{R}^{n+1}_+)$, $\mathbb{R}^{n+1}_+ := \mathbb{R}_+ \times \mathbb{R}^n$. Recall that $L_{\min}$ is the closure of $\mathcal{L}$ defined on $C_0^\infty(\mathbb{R}^{n+1}_+)$. It is known (see [4,24]) that

$$\begin{aligned}\operatorname{dom}(L_{\min}) &= H_0^{2,2}(\mathbb{R}^{n+1}_+) \\ &:= \{f \in H^{2,2}(\mathbb{R}^{n+1}_+)\colon\ f \upharpoonright \partial\mathbb{R}^{n+1}_+ = 0,\ \partial f/\partial\mathfrak{n} \upharpoonright \partial\mathbb{R}^{n+1}_+ = 0\}\end{aligned}$$

where $\mathfrak{n}$ stands for the interior normal to $\partial\mathbb{R}^{n+1}_+$. Clearly, L is symmetric. The maximal operator $L_{\max}$ is defined by $L_{\max} = (L_{\min})^*$. We emphasize that $H^{2,2}(\mathbb{R}^{n+1}_+) \subset \operatorname{dom}(L_{\max}) \subset H^{2,2}_{\mathrm{loc}}(\mathbb{R}^{n+1}_+)$ but $\operatorname{dom}(L_{\max}) \neq H^{2,2}(\mathbb{R}^{n+1}_+)$. The trace mappings $\gamma_j : C^\infty(\overline{\mathbb{R}^{n+1}_+}) \to C^\infty(\partial\mathbb{R}^{n+1}_+)$, $j \in \{0,1\}$ are defined by

$$\gamma_0 f := f \upharpoonright \partial\mathbb{R}^{n+1}_+ \quad \text{and} \quad \gamma_1 f := \partial f/\partial\mathfrak{n} \upharpoonright \partial\mathbb{R}^{n+1}_+.$$

Denote by $\mathfrak{H}_+$ the domain $\operatorname{dom}(L_{\max})$ of $L_{\max}$ equipped with the graph norm. It is known (see [24,34]) that γ_j can be extended by continuity to operators mapping $\mathfrak{H}_+$ continuously onto $H^{-j-\frac{1}{2},2}(\partial\mathbb{R}^{n+1}_+)$, $j \in \{0,1\}$.

Let us define the following realizations of $\mathcal{L}$:

(i) $L^D f := \mathcal{L}f$, $f \in \operatorname{dom}(L^D) := \{\varphi \in H^{2,2}(\mathbb{R}^{n+1}_+)\colon\ \gamma_0\varphi = 0\}$;
(ii) $L^N f := \mathcal{L}f$, $f \in \operatorname{dom}(L^N) := \{\varphi \in H^{2,2}(\mathbb{R}^{n+1}_+)\colon\ \gamma_1\varphi = 0\}$;
(iii) $L^K f := \mathcal{L}f$, $f \in \operatorname{dom}(L^K) := \{\varphi \in \operatorname{dom}(L_{\max})\colon\ \gamma_1\varphi + \Lambda\gamma_0\varphi = 0\}$ where $\Lambda := \sqrt{-\Delta_x + q(\cdot)}\colon H^{-\frac{1}{2},2}(\partial\mathbb{R}^{n+1}_+) \to H^{-\frac{3}{2},2}(\partial\mathbb{R}^{n+1}_+)$.

To treat the operator $L_{\min}$ as the Sturm–Liouville operator with (unbounded) operator potential we denote by T the minimal operator associated with the Schrödinger expression

$$-\Delta_x + q(x) := -\sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2} + q(x), \quad q = \bar{q} \in L^\infty(\mathbb{R}^n), \tag{6.1}$$

in $\mathcal{H} := L^2(\mathbb{R}^n)$. Clearly, $T = T^*$. Moreover, $T \geqslant 0$ if $q(\cdot) \geqslant 0$. Let $A := A_{\min}$ be the minimal operator associated with (1.1) where $T = T_{\min}$.

Proposition 6.1. *Let $q(\cdot) \in L^\infty(\mathbb{R})$, $q(\cdot) \geqslant 0$, and let T be the minimal (self-adjoint) operator associated with the Schrödinger expression* (6.1) *in $L^2(\mathbb{R}^n)$. Let also $t_0 := \inf \sigma(T)$ and $t_1 := \inf \sigma_{\mathrm{ess}}(T)$. Then the following hold:*

(i) *The minimal operator A coincides with the minimal operator L and* $\mathrm{dom}(A) = H_0^{2,2}(\mathbb{R}_+^{n+1})$.
(ii) *The Dirichlet realization A^D coincides with L^D, hence, L^D is absolutely continuous and $\sigma(L^D) = \sigma_{ac}(L^D) = [t_0, \infty)$.*
(iii) *The Neumann realization A^N coincides with L^N, hence, L^N is absolutely continuous and $\sigma(A^N) = \sigma_{ac}(A^N) = [t_0, \infty)$.*
(iv) *The Krein realization A^K coincides with L^K, in particular, L^K admits the decomposition $L^K = 0_{\mathcal{H}_0} \oplus (L^K)^{ac}$, $\mathcal{H}_0 := \ker(L^K)$, and $\sigma_{ac}(L^K) = [t_0, \infty)$.*
(v) *The self-adjoint realizations L^D, L^N, and L^K are ac-minimal, in particular, L^D, L^N, and $(L^K)^{ac}$ are unitarily equivalent. If $t_0 = t_1$, then the realizations L^D, L^N and L^K are strictly ac-minimal.*
(vi) *If $\widetilde{L}$ is a self-adjoint realization of $\mathcal{L}$ such that either $(\widetilde{L} - i)^{-1} - (L^D - i)^{-1} \in \mathfrak{S}_\infty(L^2(\mathbb{R}_+^{n+1}))$ or $(\widetilde{L} - i)^{-1} - (L^K - i)^{-1} \in \mathfrak{S}_\infty(L^2(\mathbb{R}_+^{n+1}))$, then $\widetilde{L}^{ac}$ and L^D are unitarily equivalent.*
(vii) *Let $\widetilde{L}$ be a self-adjoint realization of $\mathcal{L}$ such that $(\widetilde{L} - i)^{-1} - (L^N - i)^{-1} \in \mathfrak{S}_\infty(L^2(\mathbb{R}_+^{n+1}))$. If $t_0 = t_1$, then $\widetilde{L}^{ac}$ and L^D are unitarily equivalent.*

Proof. (i) We put

$$\mathcal{D}_\infty := \left\{ \sum_{1 \leqslant j \leqslant k} \phi_j(x) h_j(\xi) \colon \phi_j \in C_0^\infty(\mathbb{R}_+),\ h_j \in C_0^\infty\bigl(\mathbb{R}^n\bigr),\ k \in \mathbb{N} \right\}$$

and note that $\mathcal{D}_\infty \subseteq \mathcal{D}_0$, where $\mathcal{D}_0$ is defined by (1.2), and $\mathcal{D}_\infty \subseteq C_0^\infty(\mathbb{R}_+^{n+1})$. Moreover, $A \restriction \mathcal{D}_\infty = L \restriction \mathcal{D}_\infty$. Since $\mathcal{D}_\infty$ is a core for both minimal operators A and L, we have $A = L$ which yields $\mathrm{dom}(A) = H_0^{2,2}(\mathbb{R}_+^{n+1})$.

(ii) Since $A = L$ one has $A^F = L^F$. Moreover, $L^F = L^D$, thus $A^D = L^D$. It remains to apply Theorem 5.7(i).

(iii) Clearly, $W_T^{2,2}(\mathbb{R}_+, \mathcal{H}) = H^{2,2}(\mathbb{R}_+^{n+1})$ algebraically and topologically. A straightforward computation shows that

$$\mathfrak{t}^{\mathcal{L}}[f] := (\mathcal{L}f, f)_{\mathfrak{H}} = (\mathcal{A}f, f)_{L^2(\mathbb{R}_+^{n+1})} =: \mathfrak{t}^{\mathcal{A}}[f], \quad f \in W_T^{2,2}(\mathbb{R}_+, \mathcal{H}) = H^{2,2}\bigl(\mathbb{R}_+^{n+1}\bigr).$$

Since $W_T^{2,2}(\mathbb{R}_+, \mathcal{H})$ is dense in $W_{\sqrt{T}}^{1,2}(\mathbb{R}_+, \mathcal{H})$, the closure of $\mathfrak{t}^{\mathcal{A}}$ coincides with the quadratic form $\mathfrak{t}_N$ associated with the realization A^N and defined by (5.6). Moreover, since $H^{2,2}(\mathbb{R}_+^{n+1})$ is dense in $H^{1,2}(\mathbb{R}_+^{n+1})$, the closure of $\mathfrak{t}^{\mathcal{L}}$ is the closed quadratic form associated with L^N. Since $\mathfrak{t}^{\mathcal{A}} = \mathfrak{t}^{\mathcal{L}}$, on $W_T^{2,2}(\mathbb{R}_+, \mathcal{H})$, their closures coincide too, hence $A^N = L^N$. The remaining part follows from Theorem 5.7(ii).

(iv) Since $A = L$, the realization A^K of $\mathcal{A}$ is identical with the Krein realization L^K of $\mathcal{L}$. However, it is proved in [13, Section 9.7] that L^K is the Krein realization of $\mathcal{L}$. The remaining statements are implied by Theorem 5.7(iii).

(v) By Theorem 5.8(ii) the extensions A^D, A^N and A^K are *ac*-minimal. Combining this fact with statements (i)–(iv) we find that L^D, L^N and L^K are *ac*-minimal too. The second statement follows from Corollary 5.9(i).

(vi) Due to (i)–(iv), this statement is immediate from Theorem 5.8(iii).

(vii) This statement follows from Corollary 5.9(ii). □

Corollary 6.2. *Let the assumptions of Proposition* 6.1 *be satisfied. If*

$$\lim_{|x| \to \infty} \int_{|x-y| \leqslant 1} q(y)\,dy = 0, \tag{6.2}$$

then the realizations L^D, L^N and L^K are strictly ac-minimal and

$$\sigma\left(L^D\right) = \sigma_{ac}\left(L^K\right) = \sigma\left(L^N\right) = \sigma_{ac}\left(L^N\right) = [0, \infty). \tag{6.3}$$

In particular, conclusion (6.3) *holds whenever* $\lim_{|x|\to\infty} q(x) = 0$.

Proof. By [18, Section 60] condition (6.2) yields the equality $\sigma_c(T) = \mathbb{R}_+$, in particular $0 \in \sigma_c(T)$ and $t_1 = 0$. Since $q \geqslant 0$, we have $0 \leqslant t_0 \leqslant t_1 = 0$, that is $t_0 = t_1 = 0$. It remains to apply Proposition 6.1(i)–(iv). $\square$

Remark 6.3. (i) In Appendix A.2 we extend Proposition 6.1 for special realizations admitting separation of variables (which include the Dirichlet and Neumann realizations) of more general differential expression

$$\mathcal{L} := -\frac{\partial^2}{\partial t^2} - \Delta_x + p(t) + q(x) \tag{6.4}$$

defined in $L^2(\mathbb{R}^{n+1}_+)$. We show that these realizations are always absolutely continuous for a broad class of potentials p and q.

(ii) Let T be the (closed) minimal non-negative operator associated in $\mathcal{H} := L^2(\mathbb{R}^n)$ with general uniformly elliptic operator

$$-\sum_{j,k=1}^{n} \frac{\partial}{\partial x_j} a_{jk}(x) \frac{\partial}{\partial x_j} + q(x), \quad a_{jk} \in C^1\left(\overline{\mathbb{R}}^{n+1}_+\right),\ q \in C\left(\overline{\mathbb{R}}^{n+1}_+\right) \cap L^\infty\left(\mathbb{R}^{n+1}_+\right),$$

where the coefficients $a_{jk}(\cdot)$ are bounded with their C^1-derivatives, $q \geqslant 0$. In this case $\operatorname{dom}(T) = H^{2,2}(\mathbb{R}^n)$ algebraically and topologically. By Lemma 5.1, $\operatorname{dom}(A_{\min}) = W^{2,2}_{0,T}(\mathbb{R}_+, \mathcal{H}) = H^{2,2}_0(\mathbb{R}^{n+1}_+)$ and Proposition 6.1 remains valid with T in place of the Schrödinger operator (6.1).

Acknowledgments

We express our gratitude to the referee for extremely careful reading of the manuscript, numerous useful suggestions and valuable remarks. Following referee's advice and for the reader's convenience we added Examples 3.1, 3.2, 3.8 and Appendices B, C and D.

Appendix A. Operators admitting separation of variables

A.1. Finite interval

Here we consider the differential expression $\mathcal{A}$ with unbounded $T = T^* \geqslant 0$ (cf. (1.1)) on a finite interval $I = [0, \pi]$ and denote it by $\mathcal{A}_I$. The minimal operator $A := A_{I,\min} := \overline{A'}$ generated by $\mathcal{A}$ in the Hilbert space $\mathfrak{H}_I := L^2(I, \mathcal{H})$ is defined similarly to that of $A = A_{\min}$ in $L^2(\mathbb{R}_+, \mathcal{H})$. Obviously, $A_{I,\min}$ is densely defined and non-negative.

We briefly discuss the spectral properties of realizations of $\mathcal{A}_I$ which admit separating of variables. We set

$$A^D_I f := \mathcal{A}_I f, \quad f \in \operatorname{dom}\left(A^D_I\right) := \left\{f \in W^{2,2}_T(I, \mathcal{H}):\ f(0) = f(\pi) = 0\right\},$$
$$A^N_I f := \mathcal{A}_I f, \quad f \in \operatorname{dom}\left(A^D_I\right) := \left\{f \in W^{2,2}_T(I, \mathcal{H}):\ f'(0) = f'(\pi) = 0\right\}$$

where $W^{2,2}_T(I, \mathcal{H}) = W^{2,2}(I, \mathcal{H}) \cap L^2(I, \mathcal{H}_1(T))$ with $\mathcal{H}_1(T)$ defined by (5.1).

To state the main result denote by l_D and l_N the Dirichlet and Neumann realizations of the differential expression $l := -d^2/dx^2$ in the Hilbert space $L^2(I)$, i.e.

$$l_D := -\frac{d^2}{dx^2} \upharpoonright \mathrm{dom}(l_D), \quad \mathrm{dom}(l_D) = \bigl\{f \in W^{2,2}[0,\pi]\colon\ f(0) = f(\pi) = 0\bigr\},$$
$$l_N := -\frac{d^2}{dx^2} \upharpoonright \mathrm{dom}(l_N), \quad \mathrm{dom}(l_N) = \bigl\{f \in W^{2,2}[0,\pi]\colon\ f'(0) = f'(\pi) = 0\bigr\}.$$

Obviously, both spectra are discrete and given by $\sigma(l_D) = \{1, 4, \ldots, k^2, \ldots\}$, $k \in \mathbb{N}$ and $\sigma(l_N) = \{0, 1, 4, \ldots, k^2, \ldots\}$, $k \in \mathbb{N}_0 := \{0\} \cup \mathbb{N}$.

Proposition A.1. *Let A_I^D and A_I^N be the Dirichlet and the Neumann realizations of $\mathcal{A}_I$ in $L^2(I, \mathcal{H})$ and let $T_k := T + k^2 I_{\mathcal{H}} (\in \mathcal{C}(\mathcal{H}))$. Then*:

(i) *A_I^D is unitarily equivalent to the operator $\bigoplus_{k=1}^{\infty} T_k$.*
(ii) *A_I^N is unitarily equivalent to the operator $\bigoplus_{k=0}^{\infty} T_k$.*
(iii) *The spectrum of the operators A_I^D and A_I^N is discrete, pure point, purely singular and absolutely continuous if and only if the spectrum of T is so.*
(iv) *The spectral multiplicity functions $N_{A_I^D}(\cdot)$ and $N_{A_I^N}(\cdot)$ of the realizations A_I^D and A_I^N, respectively, are finite for each $\lambda \in \mathbb{R}$ whenever the multiplicity function $N_T(\cdot)$ is finite. Moreover, if $\sigma_{ac}(T) = [t_0, \infty)$, then $\sigma_{ac}(A_I^D) = [t_0 + 1, \infty)$ and*

$$N_{(A_I^D)^{ac}}(t) = k N_{T^{ac}}(t) \quad \textit{for a.e. } t \in \bigl[t_0 + k^2, t_0 + (k+1)^2\bigr),\ k \in \mathbb{N},$$

as well as $\sigma_{ac}(A_I^D) = [t_0, \infty)$ and

$$N_{(A_I^N)^{ac}}(t) = (k+1) N_{T^{ac}}(t) \quad \textit{for a.e. } t \in \bigl[t_0 + k^2, t_0 + (k+1)^2\bigr),$$

$k \in \mathbb{N}_0 := \{0\} \cup \mathbb{N}$.
(v) *The operators $(A_I^D)^{ac}$ and $(A_I^N)^{ac}$ are not unitarily equivalent.*

Proof. (i) By the spectral theorem, the operator $l_D = l_D^*$ is unitarily equivalent to the diagonal operator $\Lambda_D = \mathrm{diag}(1^2, 2^2, \ldots, k^2, \ldots)$ acting in $\mathfrak{H}_D = l^2(\mathbb{N})$. Namely, $U_D l_D = \Lambda_D U_D$ where U_D is the unitary map from $L^2[0,\pi]$ onto $l^2(\mathbb{N})$,

$$U_D : f = \sqrt{\frac{2}{\pi}} \sum_{k=1}^{\infty} a_k \sin kx \to \{a_k\}_1^{\infty} \in l^2(\mathbb{N})$$

and $a_k = (f, \sqrt{2/\pi} \sin kx)$. Hence

$$\begin{aligned}(U_D \otimes I_{\mathcal{H}}) A^D \bigl(U_D^* \otimes I_{\mathcal{H}}\bigr) &= (U_D \otimes I_{\mathcal{H}})(l_D \otimes I_{\mathcal{H}} + I_{\mathfrak{H}_1} \otimes T)\bigl(U_D^* \otimes I_{\mathcal{H}}\bigr)\\ &= \Lambda_D \otimes I_{\mathcal{H}} + I_{\mathfrak{H}_2} \otimes T = \bigoplus_{k=1}^{\infty} \bigl(k^2 I_{\mathcal{H}} + T\bigr) = \bigoplus_{k=1}^{\infty} T_k.\end{aligned}$$

(ii) In this case, by the spectral theorem, the operator A^N is unitarily equivalent to the diagonal operator $\Lambda_N = \mathrm{diag}(0, 1^2, 2^2, \ldots, k^2, \ldots)$ in $\mathfrak{H}_N = l^2(\mathbb{N}_0)$, $U_N l_N = \Lambda_N U_N$ where

$$U_N : f = \frac{1}{\sqrt{\pi}} b_0 + \sqrt{\frac{2}{\pi}} \sum_{k=1}^{\infty} b_k \cos kx \to \{b_k\}_0^{\infty} \in l^2(\mathbb{N}_0)$$

and $b_k = (f, \sqrt{2/\pi} \cos kx)$. Repeating the previous reasonings we arrive at the required relation

$$(U_N \otimes I_{\mathcal{H}}) A^N \big(U_N^* \otimes I_{\mathcal{H}}\big) = \bigoplus_{k=0}^{\infty} T_k.$$

(iii) This statement follows immediately from (i) and (ii) in view of the obvious relations $\sigma(\bigoplus_{k=1}^{\infty} T_k) = \bigcup_{k=1}^{\infty} \sigma(T_k)$ and $\sigma_\tau(\bigoplus_{k=1}^{\infty} T_k) = \bigcup_{k=1}^{\infty} \sigma_\tau(T_k)$, $\tau = pp, s, sc, ac$.

(iv) Combining (i) and (ii) with the obvious relations $\sigma_\tau(T_k) = k^2 + \sigma_\tau(T_k)$, $\tau = d, pp, s, sc, ac$, $k \in \mathbb{N}$ we get the statement.

(v) It follows from (i) and (ii) that $\sigma_{ac}(A_I^N) = \bigcup_{k=0}^{\infty} \sigma_{ac}(T_k)$ and $\sigma_{ac}(A_I^D) = \bigcup_{k=1}^{\infty} \sigma_{ac}(T_k)$ which yields $\sigma_{ac}(A_I^N) \neq \sigma_{ac}(A_I^D)$. This proves (v). $\square$

A.2. Semi-axis

Our next purpose is to show that the spectral properties of realizations of $\mathcal{A}$ admitting separation of variables can be investigated directly by applying elementary methods. In particular, we present a simple proof of Theorem 5.7(ii). Let us first prove a general statement.

Proposition A.2. *Let K and T be self-adjoint operators in the separable Hilbert spaces $\mathcal{K}$ and $\mathcal{H}$, respectively, and let $L := K \otimes I_{\mathcal{H}} + I_{\mathcal{K}} \otimes T$.*

(i) *Let $\mathcal{D}_K$ and $\mathcal{D}_T$ be cores of K and T, respectively. Then*

$$\mathcal{D} := \left\{ f = \sum_{1 \leqslant j \leqslant k} \phi_j \otimes h_j,\ \phi_j \in \mathcal{D}_K,\ h_j \in \mathcal{D}_T,\ k \in \mathbb{N} \right\}$$

is a core of L and L is essentially self-adjoint on $\mathcal{D}$.

(ii) $\sigma(L) = \overline{\sigma(K) + \sigma(T)}$.

(iii) *If the self-adjoint operators K and K' are unitarily equivalent, then L and $L' := K' \otimes I_{\mathcal{H}} + I_{\mathcal{K}} \otimes T$ are unitarily equivalent in $\mathfrak{H} := \mathcal{K} \otimes \mathcal{H}$.*

(iv) *If K is absolutely continuous, then L is absolutely continuous.*

Proof. (i) and (ii) are proved in [39, Theorem XIII.33].

(iii) Let V be a unitary operator in $\mathcal{K}$ such that $K' = V^* K V$. Then $U := V \otimes I_{\mathcal{H}}$ is unitary in $\mathfrak{H} = \mathcal{K} \otimes \mathcal{H}$ and

$$U^* L U = V^* \otimes I_{\mathcal{H}} (K \otimes I_{\mathcal{H}} + I_{\mathcal{K}} \otimes T) V \otimes I_{\mathcal{H}} = K' \otimes I_{\mathcal{H}} + I_{\mathcal{K}} \otimes T = L'.$$

(iv) We note that the set

$$\mathcal{M}(L) := \big\{ f \in \mathfrak{H}\colon \big(e^{-itL} f, f\big) \in L^2(\mathbb{R}) \big\}$$

is linear and dense in $\mathfrak{H}^{ac}(L)$, cf. [3, Section 3.5.2]. Let $L := K \otimes I_{\mathcal{H}} + I_{\mathcal{K}} \otimes T$. From [43, Theorem 8.35] we get

$$\big(e^{-itL} f \otimes g, f \otimes g\big) = \big(e^{-itK} f, f\big)\big(e^{-itT} g, g\big), \quad t \in \mathbb{R}.$$

If $f \in \mathcal{M}(K)$, then $f \otimes g \in \mathcal{M}(L)$ for any $g \in \mathcal{H}$ due to the obvious inclusion $(e^{-itT}g, g) \in L^\infty(\mathbb{R})$. Since $\mathcal{M}(L)$ is a linear set, we find the inclusion $\mathcal{D} := \{\sum_{1\leqslant j\leqslant k} f_j \otimes g_j\colon\ f_j \in \mathcal{M}(K),\ \ g_j \in \mathcal{H},\ k \in \mathbb{N}\} \subseteq \mathcal{M}(L)$. Taking into account that $\mathcal{M}(K)$ is dense in $\mathcal{K}$ we obtain that $\mathcal{D}$ is dense in $\mathfrak{H}$. Thus, L is absolutely continuous. $\square$

We apply Proposition A.2 to special realizations of the differential expression (6.4). To this end we consider the differential expression $l'(p) := -\frac{d^2}{dt^2} + p$ in $\mathcal{K} := L^2(\mathbb{R}_+)$ assuming that $p = \overline{p} \in L^2_{\mathrm{loc}}(\overline{\mathbb{R}}_+)$. Clearly, it is well defined on $\mathrm{dom}(l'(p)) := C_0^\infty(\mathbb{R}_+)$. Its closure, the minimal symmetric operator, is denoted by $l(p)$. In what follows we assume that $l(p)$ is in the limit point case at infinity. It is known that the maximal operator $l(p)^*$ has the regularity property: $\mathrm{dom}(l(p)^*) \supset W^{2,2}_{\mathrm{loc}}(\mathbb{R}_+)$. Therefore the following extensions are well defined

$$l_\tau(p) := \left(-\frac{d^2}{dt^2} + p\right) \upharpoonright \big\{f \in \mathrm{dom}\big(l(p)^*\big)\colon\ f'(0) = \tau f(0)\big\}$$

and self-adjoint in $\mathcal{K} := L^2(\mathbb{R}_+)$ for $\tau \in \mathbb{R} \cup \{0\} \cup \{\infty\}$. For $\tau = 0$ and $\tau = \infty$ these extensions are identified with the Neumann and the Dirichlet realizations of $-\frac{d^2}{dt^2} + p$, respectively.

Further, let $T = T^* \geqslant 0$, $T \in \mathcal{C}(\mathcal{H})$. Consider the symmetric operator $A'(p) := l(p) \otimes I_{\mathcal{H}} + I_{\mathcal{K}} \otimes T$ defined in $\mathfrak{H} := \mathcal{K} \otimes \mathcal{H} := L^2(\mathbb{R}_+, \mathcal{H})$ on the domain

$$\mathrm{dom}\big(A'(p)\big) := \left\{f = \sum_{1\leqslant j\leqslant k} \phi_j h_j\colon\ \phi_j \in C_0^\infty(\mathbb{R}_+),\ h_j \in \mathrm{dom}(T),\ k \in \mathbb{N}\right\},$$

cf. (1.1) and (1.2). The closure $A(p)$ of $A'(p)$ is the minimal operator associated with (6.4). Let $A_\tau(p) := l_\tau(p) \otimes I_{\mathcal{H}} + I_{\mathcal{K}} \otimes T$. By Proposition A.2(i), $A_\tau(p)$ is a self-adjoint extension of $A(p)$ in $\mathfrak{H}$. In particular, $A_0 = A^N$ and $A_\infty = A^D$.

Corollary A.3. *Let $T = T^* \geqslant 0$. Further, let $p \in L^2_{\mathrm{loc}}(\overline{\mathbb{R}}_+)$ and let $l(p)$ be in the limit point case at infinity.*

(i) *If the self-adjoint extensions $l_{\tau_1}(p)$ and $l_{\tau_2}(p)$ of $l(p)$ are unitarily equivalent, then the self-adjoint operators $A_{\tau_1}(p) := l_{\tau_1}(p) \otimes I_{\mathcal{H}} + I_{\mathcal{K}} \otimes T$ and $A_{\tau_2}(p) := l_{\tau_2}(p) \otimes I_{\mathcal{H}} + I_{\mathcal{K}} \otimes T$ are unitarily equivalent.*
(ii) *If the self-adjoint extension $l_\tau(p)$ is absolutely continuous, then $A_\tau(p)$ is absolutely continuous too.*
(iii) *Assume that $p(t) \geqslant 0$ and either $\int_0^\infty tp(t)\,dt < \infty$ or $\int_0^\infty p(t)\,dt < \infty$. Then for any $\tau \in \mathbb{R}_+$ the operator $A_\tau(p)$ is unitarily equivalent to the Dirichlet realization $A_\infty(p) = A^D(p)$. In particular, all extensions $A_\tau(p)$ are absolutely continuous.*

Proof. (i) is immediate from Proposition A.2(iii).

(ii) follows from Proposition A.2(iv).

(iii) It is well known that under the assumptions on p any two non-negative self-adjoint extensions $l_{\tau_1}(p)$ and $l_{\tau_2}(p)$ are unitarily equivalent and purely absolutely continuous. One completes the proof by applying (i) and (ii). $\square$

Let $T' = -\Delta + q$ be the Schrödinger expression with $q \in L^2_{\mathrm{loc}}(\mathbb{R}^n)$. Assume that T' is essentially self-adjoint on $C_0^\infty(\mathbb{R}^n)$ and denote by T its closure. Consider the symmetric operator $L''(p) := l'(p) \otimes I_{\mathcal{H}} + I_{\mathcal{K}} \otimes T'$ defined on

$$\mathrm{dom}\big(L''(p)\big) := \mathcal{D} := \left\{f = \sum_{1\leqslant j\leqslant k} \phi_j h_j\colon\ \phi_j \in C_0^\infty(\mathbb{R}_+),\ h_j \in C_0^\infty\big(\mathbb{R}^n\big),\ k \in \mathbb{N}\right\}.$$

Clearly, $L''(p) \subseteq L'(p) \subseteq \overline{L''(p)}$ where $L'(p) = A'(p)$. Hence $\overline{L''(p)} = \overline{L'(p)} = L(p)$ and any self-adjoint extension $\widetilde{L}(p)$ of $L(p)$ is a self-adjoint extension of

$$\mathcal{L}(p) := -\frac{\partial^2}{\partial t^2} - \Delta_x + p(t) + q(x) \tag{A.1}$$

defined on $C_0^\infty(\mathbb{R}_+^{n+1})$. By Corollary A.3 a self-adjoint realization $L_\tau(p) = l_\tau(p) \otimes I_{\mathcal{H}} + I_{\mathcal{K}} \otimes T$ of $\mathcal{L}(p)$ is absolutely continuous whenever the self-adjoint extension $l_\tau(p)$ of $l(p)$ is absolutely continuous. In particular, this is the case provided that $p(t) \geqslant 0$ for $t \in \mathbb{R}_+$ and either $\int_0^\infty tp(t)\,dt < \infty$ or $\int_0^\infty p(t)\,dt < \infty$.

Corollary A.3 can also be applied to Schrödinger operator (A.1) defined in $L^2(\mathbb{R}^{n+1})$. Indeed, let $p \in L^2_{\mathrm{loc}}(\mathbb{R})$ and let the differential operator $l'(p) = -\frac{d^2}{dt^2} + p(t)$ be essentially self-adjoint on $C_0^\infty(\mathbb{R})$. If its closure is absolutely continuous, then, by Proposition A.2, the Schrödinger operator $\mathcal{L}(p)$ defined on $C_0^\infty(\mathbb{R}^{n+1})$ is essential self-adjoint in $L^2(\mathbb{R}^{n+1})$ and its closure is absolutely continuous, too.

Remark A.4. (i) We note that the above reasonings cannot be applied to realizations of $\mathcal{A}$ which do not admit a tensor structure $\widetilde{A} = \widetilde{l} \otimes I_{\mathcal{H}} + I_{\mathcal{K}} \otimes T$.

(ii) Comparing Corollary A.3 with Proposition A.1 we see that the spectral properties of realizations of $\mathcal{A}$ on the semi-axis $\mathbb{R}_+$ substantially differ from that of realizations of $\mathcal{A}$ on a finite interval I. Indeed, for self-adjoint realizations of $\mathcal{A}$ on $\mathbb{R}_+$ the *ac*-part can never be eliminated for any $T = T^* \geqslant 0$, cf. Theorem 5.8(ii). In contrast to that the spectral properties of self-adjoint realizations of $\mathcal{A}_I$ strongly depend on T.

Appendix B. Spectral multiplicity function

The classical definition of the spectral multiplicity function can be found, for instance, in [7, Section 7], and is in detail analyzed there. In the present paper we use another definition of this notion proposed in [35]. This definition is also applied to non-orthogonal measures. By $\mathcal{B}(\mathbb{R})$ we denote the σ-algebra of Borel subsets of $\mathbb{R}$. The set of bounded Borel subsets is denoted by $\mathcal{B}_b(\mathbb{R})$.

Definition B.1. (See [35, Definition 2.1].) A mapping $\Sigma : \mathcal{B}_b(\mathbb{R}) \to [\mathfrak{H}]$ is an operator measure if the following conditions are satisfied:

(i) $\Sigma(\emptyset) = 0$,
(ii) $\Sigma(\delta) = \Sigma(\delta)^* \geqslant 0$ for $\delta \in \mathcal{B}_b(\mathbb{R})$,
(iii) the function is strongly countably additive, i.e., if $\delta = \bigcup_{j=1}^\infty \delta_j$ is a union of disjoint Borel sets $\delta_j \in \mathcal{B}_b(\mathbb{R})$, then $\Sigma(\delta) = \text{s-lim}_{N\to\infty} \Sigma(\bigcup_{j=1}^N \delta_j)$.

The operator measure Σ is called bounded if it is defined on the $\mathcal{B}(\mathbb{R})$ and $\Sigma(\mathbb{R}) \in [\mathfrak{H}]$.

The bounded operator measure is called spectral or orthogonal if $\Sigma(\delta)$ is an orthogonal projection for $\delta \in \mathcal{B}(\mathbb{R})$, that is $\Sigma(\delta)^2 = \Sigma(\delta)$.

An operator measure Σ_1 is called absolutely continuous with respect to an operator measure Σ_2 if $\Sigma_1(\delta) = 0$ whenever $\Sigma_2(\delta) = 0$. The operator measures Σ_1 and Σ_2 are called equivalent if they are mutually absolutely continuous. There always exists a scalar measure ϱ equivalent to the operator measure Σ.

Definition B.2. (See [35, Definition 4.5].) Let Σ be an operator measure and let ϱ be an equivalent scalar measure. Further, let $\{e_j\}_{j=1}^d$, $d = \dim(\mathfrak{H})$, be an orthonormal basis in $\mathfrak{H}$. Let

$$\sigma_{ij}(\delta) := \big(\Sigma(\delta)e_i, e_j\big), \qquad \psi_{ij}(t) = \frac{d\sigma_{ij}}{d\varrho}(t),$$

$$\Psi_n^e(t) := \big(\psi_{ij}(t)\big)_{i,j=1}^n, \qquad \Psi^e(t) := \big(\psi_{ij}(t)\big)_{i,j=1}^d,$$

where $\delta \in \mathcal{B}_b(\mathbb{R})$, $t \in \mathbb{R}$ and $1 \leqslant n \leqslant d$. The function

$$N^e_\Sigma(t) := \operatorname{rank}\bigl(\Psi^e(t)\bigr) := \sup_{n\geqslant 1} \operatorname{rank}\bigl(\Psi^e_n(t)\bigr)$$

is called the spectral multiplicity function, where $\operatorname{rank}(\cdot)$ denotes the rank of a matrix.

Remark B.3. (i) The derivative $\frac{d\sigma_{ij}}{d\varrho}$ means the Radon–Nikodym derivative. It is defined only for $\varrho(\cdot)$-a.e. $t \in \mathbb{R}$.

(ii) In fact, the definition of the spectral multiplicity function $N^e_\Sigma(\cdot)$ does not depend on the choice of the orthonormal basis $e = \{e_j\}_{j=1}^d$ (see [35, Proposition 4.6]). Therefore we omit the upper index e and write $N_\Sigma(\cdot)$ instead of $N^e_\Sigma(\cdot)$.

(iii) For orthogonal bounded operator measures this definition is equivalent to the classical one, cf. [7, Section 7].

(iv) If $T = T^* \in \mathcal{C}(\mathfrak{H})$, then the spectral multiplicity function $N_T(\cdot)$ of T is defined by $N_T(t) := N_{E_T}(t)$, $t \in \mathbb{R}$, where $E_T(\cdot)$ is the spectral measure of T.

To compute the spectral multiplicity function we consider the operator measure $\Sigma_D(\delta) := D^*\Sigma(\delta)D$, $\delta \in \mathcal{B}_b(\mathbb{R})$, where $D \in \mathfrak{S}_2(\mathcal{H})$. Clearly, that $\Sigma_D(\cdot)$ is equivalent to the scalar measure $\varrho(\cdot) = \operatorname{tr}(\Sigma_D(\cdot))$. Since $\Sigma_D(\cdot)$ takes values in $\mathfrak{S}_1(\mathcal{H})$ the Radon–Nikodym derivative $\frac{d\Sigma_D}{d\varrho}$ exists. We set $\Psi_{\Sigma_D}(t) := \frac{d\Sigma_D}{d\varrho}(t)$ for ϱ-a.e. $t \in \mathbb{R}$. If $\ker(D) = \ker(D^*) = \{0\}$ one gets from [35, Corollary 4.7]

$$N_\Sigma(t) = N_{\Sigma_D}(t) := \dim\bigl(\overline{\operatorname{ran}\bigl(\Psi_{\Sigma_D}(t)\bigr)}\bigr) \quad \text{for } \varrho\text{-a.e. } t \in \mathbb{R}.$$

The computation is simplified if the operator measure Σ admits already the representation

$$\Sigma(\delta) = \int_\delta \Psi_\Sigma(t)\, d\varrho(t), \quad \delta \in \mathcal{B}_b(\mathbb{R}).$$

In this case we set $N_\Sigma(t) := \dim(\overline{\operatorname{ran}(\Psi_\Sigma(t))})$ for ϱ-a.e. $t \in \mathbb{R}$.

Any operator measure admits the unique Lebesgue decomposition

$$\Sigma = \Sigma^s + \Sigma^{ac}$$

where the operator measure Σ^{ac} is absolutely continuous with respect to the Lebesgue measure. In what follows $N_{\Sigma^{ac}}(\cdot)$ is called the spectral multiplicity function of the *ac*-part of Σ.

In [38] the spectral multiplicity function of the absolutely continuous part of Σ is related to an R-function or Nevanlinna function. A Nevanlinna function $F : \mathbb{C}_+ \to [\mathfrak{H}]$ is an operator-valued function holomorphic in the upper half plane $\mathbb{C}_+$ and satisfying $\operatorname{Im}(F(z)) := \frac{1}{2i}(F(z) - F(z)^*) \geqslant 0$, $z \in \mathbb{C}_+$. Every R-function admits the integral representation

$$F(z) = C_0 + C_1 z + \int_\mathbb{R} \left(\frac{1}{t-z} - \frac{t}{1+t^2}\right) d\Sigma_F(t), \quad z \in \mathbb{C}_+,$$

where C_0, C_1 are bounded self-adjoint operators, $C_1 \geqslant 0$ and $\Sigma_F(\cdot)$ is an operator-valued measure defined on $\mathcal{B}_b(\mathbb{R})$ and satisfying $\int_\mathbb{R} \frac{1}{1+t^2}\, d\Sigma_F(t) \in [\mathfrak{H}]$.

To compute $N_{\Sigma^{ac}_F}(\cdot)$ let us consider the sandwiched Nevanlinna function $F^D(\cdot) := D^*F(\cdot)D$ where $D \in \mathfrak{S}_2(\mathcal{H})$ and satisfies $\ker(D) = \ker(D^*) = \{0\}$. In this case the limits $F^D(t) := \lim_{y\to+0} F^D(t+iy)$ exist for a.e. $t \in \mathbb{R}$ in the Hilbert–Schmidt norm [6,17] and

$$N_{\Sigma_F^{ac}}(t) = d_{F^D}(t) := \dim\bigl(\mathrm{ran}\bigl(\mathrm{Im}\bigl(F^D(t+i0)\bigr)\bigr)\bigr) \tag{B.1}$$

for a.e. $t \in \mathbb{R}$ with respect to the Lebesgue measures, cf. [38, Proposition 2.6]. In fact, if the weak limits $F(t) := \text{w-lim}_{y\to+0} F(t+iy)$ exist for a.e. $t \in R$, then $N_{\Sigma_F^{ac}}(t) = d_{F^D}(t) := \dim(\mathrm{ran}(\mathrm{Im}(F(t+i0))))$ for a.e. $t \in R$, that is, upper index D can be omitted, cf. [38, Corollary 2.7].

Appendix C. Absolutely continuous closure

The concept of the *ac*-closure has been introduced in [10] (and independently in [15]). Its properties can also be found in [10,15]. Here we recall the definition of *ac*-closure of a Borel subset of $\mathbb{R}$ used in Section 3.

Definition C.1. (See [10].) Let $\delta \in \mathcal{B}(\mathbb{R})$. The set $\mathrm{cl}_{ac}(\delta)$ defined by

$$\mathrm{cl}_{ac}(\delta) := \bigl\{x \in \mathbb{R}\colon \bigl|(x-\varepsilon, x+\varepsilon) \cap \delta\bigr| > 0\ \forall \varepsilon > 0\bigr\}$$

is called the absolutely continuous closure (*ac*-closure) of the Borel set $\delta \in \mathcal{B}(\mathbb{R})$.

Let us recall simple properties of the absolutely continuous closure.

Lemma C.2. *(See [15,38].) Let $\delta, \delta', \delta_k \in \mathcal{B}(\mathbb{R})$, $k \in \mathbb{N}$. Then,*

(i) $\mathrm{cl}_{ac}(\delta)$ *is a closed set*;
(ii) $\mathrm{cl}_{ac}(\delta) \subseteq \bar{\delta}$;
(iii) $|\delta \setminus \mathrm{cl}_{ac}(\delta)| = 0$;
(iv) $\mathrm{cl}_{ac}(\mathrm{cl}_{ac}(\delta)) = \mathrm{cl}_{ac}(\delta)$;
(v) $|\mathrm{cl}_{ac}(\delta)| \geqslant |\delta|$;
(vi) *if* $\delta \subseteq \delta'$, *then* $\mathrm{cl}_{ac}(\delta) \subseteq \mathrm{cl}_{ac}(\delta')$;
(vii) *if* $|\delta| = 0$, *then* $\mathrm{cl}_{ac}(\delta) = \emptyset$;
(viii) *if* $|\delta'| = 0$, *then* $\mathrm{cl}_{ac}(\delta \cup \delta') = \mathrm{cl}_{ac}(\delta)$;
(ix) *if* $\delta := \bigcup_{k\in\mathbb{N}} \delta_k$, *then* $\mathrm{cl}_{ac}(\delta) = \overline{\bigcup_{k\in\mathbb{N}} \mathrm{cl}_{ac}(\delta_k)}$.

Let us indicate several examples.

Example C.3. (i) Let $\delta = [0,1] \cup x_0$ where $x_0 \notin [0,1]$. Then $\mathrm{cl}_{ac}(\delta) = [0,1]$.

(ii) Let δ be the set of rational numbers of the closed interval $[0,1]$, $\delta = [0,1] \cap \mathbb{Q}$. Then $\mathrm{cl}_{ac}(\delta) = \emptyset$ while $\bar{\delta} = [0,1]$.

(iii) [15, Example 2.12] Let $\{r_n\}_{n\in\mathbb{N}}$ be an enumeration of the rational numbers in $[0,1]$ and let

$$\delta := \bigcup_{n\in\mathbb{N}} \biggl(r_n - \frac{1}{4^n}, r_n + \frac{1}{4^n}\biggr).$$

Then $\mathrm{cl}_{ac}(\delta) \supseteq [0,1]$. Moreover, one has $|\mathrm{cl}_{ac}(\delta)| \geqslant 1$ and $|\delta| \leqslant \frac{2}{3}$. Note, in addition, that $|\mathrm{cl}_{ac}(\delta) \setminus \delta| \geqslant \frac{1}{3}$ and $|\delta \setminus \mathrm{cl}_{ac}(\delta)| = 0$.

Finally, we refer to Proposition 2.8, see also [38, Proposition 3.2], as a useful application of the *ac*-closure concept.

Appendix D. Linear relations

Here we briefly recall some basic facts on linear relations. Let $\mathcal{H}$ be a separable Hilbert space. A linear relation Θ in $\mathcal{H}$ is a linear (not necessarily closed) subspace of the Cartesian product $\mathcal{H} \times \mathcal{H}$. The domain, range, kernel and multivalued part are defined by

$$\mathrm{dom}(\Theta) := \{f \in \mathcal{H}\colon \{f, f'\} \in \Theta\}, \qquad \mathrm{ran}(\Theta) := \{f' \in \mathcal{H}\colon \{f, f'\} \in \Theta\},$$
$$\ker(\Theta) := \{f \in \mathcal{H}\colon \{f, 0\} \in \Theta\}, \qquad \mathrm{mul}(\Theta) := \{f' \in \mathcal{H}\colon \{0, f'\} \in \Theta\},$$

respectively. The inverse relation Θ^{-1} is given by

$$\Theta^{-1} := \{\{f', f\} \in \mathcal{H} \times \mathcal{H}\colon \{f, f'\} \in \Theta\}.$$

Obviously, one has $\mathrm{dom}(\Theta^{-1}) = \mathrm{ran}(\Theta)$, $\mathrm{ran}(\Theta^{-1}) = \mathrm{dom}(\Theta)$ and $\ker(\Theta^{-1}) = \mathrm{mul}(\Theta)$. The relation Θ is called closed if Θ is a closed subspace in $\mathcal{H} \times \mathcal{H}$. A relation Θ is always closable, its closure is denoted by $\overline{\Theta}$. If Θ is closed, then $\ker(\Theta)$, $\mathrm{mul}(\Theta)$ and Θ^{-1} are closed. The sum and the product of two linear relations Θ' and Θ'' are defined by

$$\Theta' + \Theta'' := \{\{f, f' + f''\} \in \mathcal{H} \times \mathcal{H}\colon \{f, f'\} \in \Theta',\ \{f, f''\} \in \Theta''\} \tag{D.1}$$

and

$$\Theta''\Theta' := \{\{f, f''\} \in \mathcal{H} \times \mathcal{H}\colon \{f, g\} \in \Theta',\ \{g, f''\} \in \Theta''\}.$$

The adjoint relation Θ^* is defined by

$$\Theta^* := \{\{g, g'\} \in \mathcal{H} \times \mathcal{H}\colon (f, g') = (f', g) \text{ for all } \{f, f'\} \in \Theta\}.$$

It is worth to mention that, as distinct from the operator case, the inverse relation and the adjoint relation always exist. We note that

$$\left(\mathrm{ran}(\Theta)^*\right)^{\perp} = \ker(\overline{\Theta}) \quad \text{and} \quad \left(\mathrm{dom}(\Theta^*)\right)^{\perp} = \mathrm{mul}(\overline{\Theta}).$$

A relation Θ is called symmetric if $\Theta \subseteq \Theta^*$ and self-adjoint if $\Theta = \Theta^*$. The linear relations $\Theta_0 := \mathcal{H} \times \{0\}$ and $\Theta_1 := \{0\} \times \mathcal{H}$ are closed and self-adjoint. Clearly, $\Theta_0^{-1} = \Theta_1$.

If B is a linear (not necessarily densely) operator acting in $\mathcal{H}$, then its graph $\mathrm{gr}(B) := \{\{f, Bf\}\colon f \in \mathrm{dom}(B)\}$ is a linear relation in $\mathcal{H}$. It is always identified the operator with its graph. Clearly, $\mathrm{dom}(\mathrm{gr}(B)) = \mathrm{dom}(B)$, $\mathrm{ran}(\mathrm{gr}(B)) = \mathrm{ran}(B)$ and $\ker(\mathrm{gr}(B)) = \ker(B)$.

The part $\mathrm{mul}(\mathrm{gr}(B))$ is always trivial, i.e., $\mathrm{mul}(\mathrm{gr}(B)) = \{0, 0\}$. If the inverse operator B^{-1} exists, then $\mathrm{gr}(B^{-1}) = \mathrm{gr}(B)^{-1}$. By definition, the operator B is closed if its graph $\mathrm{gr}(B)$ is closed. If the adjoint operator B^* exists, then $\mathrm{gr}(B^*) = \mathrm{gr}(B)^*$. The relation Θ_0 is the graph of the zero operator $\mathbb{O}$, $\Theta_0 = \mathrm{gr}(\mathbb{O})$, while Θ_1 is not a graph.

If Θ is a linear relation and $z \in \mathbb{C}$, then $\Theta - zI$ is well defined, cf. (D.1). It is said that z belongs to the resolvent set $\varrho(\Theta)$ of Θ, if the relation $(\Theta - zI)^{-1}$ is the graph of a bounded operator defined on $\mathcal{H}$. Clearly, if B is an operator and $z \in \varrho(B)$, then $z \in \varrho(\mathrm{gr}(B))$ and vise versa. The set $\sigma(\Theta) := \mathbb{C} \setminus \varrho(\Theta)$ is called the spectrum of the relation Θ.

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